Theorems (IVT, EVT, and MVT)

Students should be able to apply and have a geometric understanding of the following:

- Intermediate Value Theorem
- Mean Value Theorem for derivatives
- Extreme Value Theorem

Multiple Choice

1. (calculator not allowed)

   If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

   (A) $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.

   (B) $f'(c) = 0$ for some c such that $a < c < b$.

   (C) f has a minimum value on $a \le x \le b$.

   (D) f has a maximum value on $a \le x \le b$.

   (E) $\int_a^b f(x)\,dx$ exists.

2. (calculator not allowed)

   The function f is defined on the closed interval $[2,4]$ and $f(2) = f(3) = f(4)$. On the open interval $(2,4)$, f is continuous and strictly decreasing. Which of the following statements is true?

   (A) f attains neither a minimum value nor a maximum value on the closed interval $[2,4]$.

   (B) f attains a minimum value but does not attain a maximum value on the closed interval $[2,4]$.

   (C) f attains a maximum value but does not attain a minimum value on the closed interval $[2,4]$.

   (D) f attains both a minimum value and a maximum value on the closed interval $[2,4]$.

3. (calculator not allowed)

Let f be a function with first derivative defined by $f'(x)=\frac{2x^2-5}{x^2}$ for $x>0$. It is known that $f(1)=7$ and $f(5)=11$. What value of x in the open interval $(1,5)$ satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1,5]$?

(A) 1 (B) $\sqrt{\frac{5}{2}}$ (C) $\sqrt[3]{10}$ (D) $\sqrt{5}$

4. (calculator not allowed)

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x)=\frac{1}{2}$ must have at least two solutions in the interval [0, 2] if k =

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) 3

5. (calculator not allowed)

Let g be a continuous function on the closed interval [0, 1]. Let $g(0)=1$ and $g(1)=0$. Which of the following is NOT necessarily true?

(A) There exists a number h in [0, 1] such that $g(h)\geq g(x)$ for all x in [0, 1].

(B) For all a and b in [0, 1], if $a = b$, then $g(a)=g(b)$.

(C) There exists a number h in [0, 1] such that $g(h)=\frac{1}{2}$.

(D) There exists a number h in [0, 1] such that $g(h)=\frac{3}{2}$.

(E) For all h in the open interval (0, 1), $\lim_{x\to h} g(x)=g(h)$.

6. (calculator not allowed)

If $f(x) = \sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be c?

(A) $\frac{2\pi}{3}$

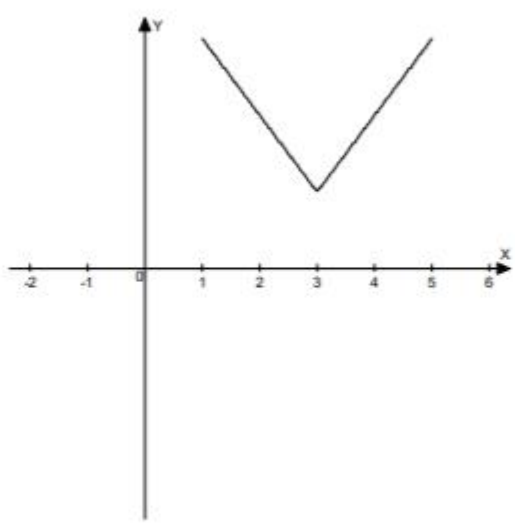
(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{6}$

(D) π

(E) $\frac{3\pi}{2}$

7. (calculator not allowed)
Which of the following theorems may be applied to the graph below, $y = |x-3| + b,\ b > 0,$ over the interval $[2,\ 4]$?



I. Mean Value Theorem
II. Intermediate Value Theorem
III. Extreme Value Theorem

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

8. (calculator not allowed)

The function f is defined by $f(x) = 4x^2 - 5x + 1$. The application of the Mean Value Theorem to f on the interval $0 < x < 2$ guarantees the existence of a value c, where $0 < c < 2$ such that $f'(c) =$

(A) 1 (B) 3 (C) 7 (D) 8

9. (calculator not allowed)

A function of f is continuous on the closed interval $[2,5]$ with $f(2)=17$ and $f(5)=17$. Which of the following additional conditions guarantees that there is a number c in the open interval $(2,5)$ such that $f'(c)=0$?

(A) No additional conditions are necessary
(B) f has a relative extremum on the open interval $(2,5)$.
(C) f is differentiable on the open interval $(2,5)$.
(D) $\int_2^5 f(x)dx$ exists

10. (calculator not allowed)

x	0	2	4	8
$f(x)$	3	4	9	13
$f'(x)$	0	1	1	2

The table above gives values of a differentiable function f and its derivatives at selected values of x. If h is the function given by $h(x) = f(2x)$, which of the following statements must be true?

(I) h is increasing on $2 < x < 4$.
(II) There exists c, where $0 < c < 4$, such that $h(c) = 12$.
(III) There exists c, where $0 < c < 2$, such that $h'(c) = 3$.

(A) II only
(B) I and III only
(C) II and III only
(D) I, II, and III

11. (calculator allowed)

The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

(A) There exists c, where $-2<c<1$, such that $f(c)=0$.

(B) There exists c, where $-2<c<1$, such that $f'(c)=0$.

(C) There exists c, where $-2<c<1$, such that $f(c)=3$.

(D) There exists c, where $-2<c<1$, such that $f'(c)=3$.

(E) There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

12. (calculator not allowed)

x	0	2	5	9	11
$g(x)$	1	2.8	1.7	1	3.4

The table above shows selcted values of a continuous function g. For $0 \le x \le 11$, what is the fewest possible number of times $g(x)=2$?

(A) One (B) Two (C) Three (D) Four

Free Response

13. (calculator not allowed)

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

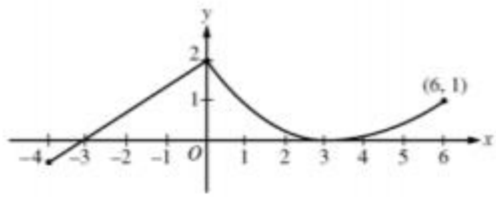
14. (calculator not allowed)

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

15. (calculator allowed)

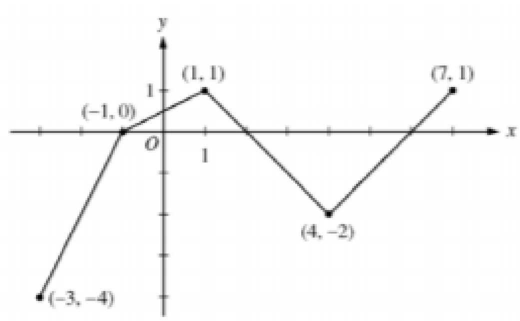


Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

16. (calculator not allowed)



Graph of g'

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

18. (calculator not allowed)

Let f be a twice-differentiable function such that $f(2)=5$ and $f(5)=2$. Let g be the function given by $g(x)=f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c)=-1$.

(b) Show that $g'(2)=g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k)=0$.

(d) Let $h(x)=f(x)-x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r)=0$.