Key

AP FRQ Review: Differential Equations

1) Non-Calculator

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$\frac{dH}{dt} = \frac{-1}{4}(H-27)$

a) find slope using $\frac{dH}{dt}$: $H'(0) = \frac{-1}{4}(91-27) = -16$ $\quad H(0)=91$

point: $(0,91)$

slope: $m=-16$

$y-91 = -16(t-0)$

$\boxed{y = -16t+91}$

$y(3) = -16(3)+91 = \boxed{43\text{° Celsius}}$

b) $\frac{dH}{dt} = \frac{-1}{4}(H-27)$

$\frac{d^2H}{dt^2} = \frac{-1}{4}\left(\frac{dH}{dt}\right)$

$\frac{d^2H}{dt^2} = \frac{-1}{4}\left[\frac{-1}{4}(H-27)\right]$

$\frac{d^2H}{dt^2} = \frac{1}{16}(H-27)$

when $H > 27°$, $\frac{d^2H}{dt^2} > 0$ (concave up)

since $y(3) > 27°$, graph of H is concave up and part (a) is an underestimate

c) $\frac{dG}{dt} = -(G-27)^{2/3}$

$dG = -(G-27)^{2/3}dt$

$\int \frac{dG}{(G-27)^{2/3}} = \int -1dt$

$\int (G-27)^{-2/3} dG = -\int 1dt$

$u = G-27$

$\frac{du}{dG} = 1$

$\int u^{-2/3}du = -\int 1dt$

$\frac{u^{1/3}}{1/3} = -1t + C$

$3(G-27)^{1/3} = -t+C$ ← solve for C $(0,91)$

$3(91-27)^{1/3} = -0+C$

$3(4) = C$

$12 = C$

$3(G-27)^{1/3} = -t+12$

$(G-27)^{1/3} = \frac{-t+12}{3}$

$G-27 = \left(\frac{-t+12}{3}\right)^3$

continued up top

...(c) continued

$G = \left(\frac{-t+12}{3}\right)^3 + 27$

$G(3) = \left(\frac{-3+12}{3}\right)^3 + 27$

$= 27+27$

$\boxed{G(3) = 54\text{° Celsius}}$

2) Non-Calculator

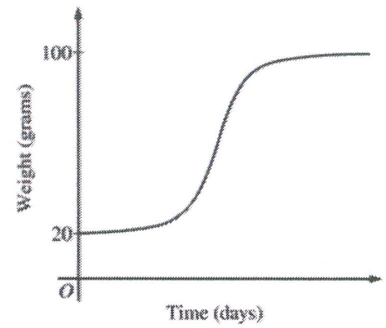
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100-40) = 12$ $\quad \left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100-70) = \frac{1}{5}(30) = 6$

*Bird is gaining weight faster when it weighs 40 grams.

b) since $\frac{dB}{dt} = \frac{1}{5}(100) - \frac{1}{5}B$, then $\frac{d^2B}{dt^2} = -\frac{1}{5}\left(\frac{dB}{dt}\right)$, $\frac{d^2B}{dt^2} = -\frac{1}{5}\left(\frac{1}{5}(100-B)\right)$

$\frac{d^2B}{dt^2} = -\frac{1}{25}(100-B)$. When $20 < B < 100$, $\frac{d^2B}{dt^2} < 0$ meaning the graph is concave down. However, the graph is showing concave up for some portions of graph.

c) $\frac{dB}{dt} = \frac{1}{5}(100-B)$

$\frac{dB}{dt} = \frac{100-B}{5}$

$5dB = (100-B)dt$

$\int \frac{dB}{100-B} = \int \frac{dt}{5}$

$u = 100-B$

$\frac{du}{dB} = -1 \quad dB = -du$

$\int \frac{1}{u}(-du) = \frac{1}{5}\int 1\,dt$

$-1\left(-\ln|u| = \frac{1}{5}t + c\right)$

$\ln|100-B| = -\frac{1}{5}t + c$

$e^{\ln|100-B|} = e^{-\frac{1}{5}t + c}$

$|100-B| = e^{-\frac{1}{5}t} \cdot e^{c}$

$100 - B = Ce^{-\frac{1}{5}t}$ find C. plug in (0,20)

$100 - 20 = Ce^{-\frac{1}{5}(0)}$

$80 = C(1)$, $C = 80$

$100 - B' = 80e^{-\frac{1}{5}t}$

$100 - B = 80e^{-\frac{1}{5}t}$

$100 - 80e^{-\frac{1}{5}t} = B$

$\boxed{B(t) = 100 - 80e^{-\frac{1}{5}t}}$

3) Non-Calculator

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

a) * use $\frac{dW}{dt}$ to find slope. $\frac{dW}{dt} = \frac{1}{25}(W-300) \rightarrow \left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(1400-300) = 44$ $w(0)=1400$

point: $(0, 1400)$
slope: $m = 44$

$y - y_1 = m(x - x_1)$
$W - W_1 = m(t - t_1)$

$W - 1400 = 44(t - 0)$
$W = 44t + 1400$

$W(\tfrac{1}{4}) = 44(\tfrac{1}{4}) + 1400$
$\boxed{W(\tfrac{1}{4}) = 1411 \text{ tons}}$

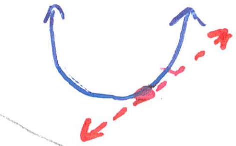
b) Since $\frac{dW}{dt} = \frac{1}{25}W - \frac{1}{25}(300)$

$\frac{d^2W}{dt^2} = \frac{1}{25}\left(\frac{dW}{dt}\right)$

$\frac{d^2W}{dt^2} = \frac{1}{25}\left(\frac{1}{25}(W-300)\right)$

$\frac{d^2W}{dt^2} = \frac{1}{625}(W-300)$

* Since $W \geq 1400$ and always increasing, $\frac{d^2W}{dt^2} > 0$, meaning $W(t)$ is concave up.
If graph is concave up, then the linear approximation will be an underestimate.



c) $\frac{dW}{dt} = \frac{W-300}{25}$

$25\,dW = (W-300)\,dt$

$\frac{dW}{W-300} = \frac{dt}{25}$

$u = W - 300$
$\frac{du}{dW} = 1$
$dW = du$

$\int \frac{du}{u} = \frac{1}{25}\int 1\,dt$

$\ln|u| = \frac{1}{25}t + C$

$\ln|W-300| = \frac{1}{25}t + C$

$e^{\ln|W-300|} = e^{\frac{1}{25}t + C}$

$|W-300| = e^{\frac{1}{25}t} \cdot e^C$

$W - 300 = Ce^{\frac{1}{25}t}$

$W = Ce^{\frac{1}{25}t} + 300$ ← solve for C $(0, 1400)$

$1400 = Ce^0 + 300$

$1400 - 300 = C$

$C = 1100$

$\boxed{W(t) = 1100e^{\frac{1}{25}t} + 300}$

4) Non-Calculator

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

a) find slope using $\frac{dy}{dx} = xy^3 \rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = (1)(2)^3 = 8$

point: $(1,2)$

slope: $m = 8$

$y - y_1 = m(x - x_1)$

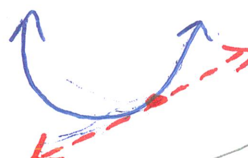
$\boxed{y - 2 = 8(x-1)}$

b) $y = 8(x-1) + 2$

$y(1.1) \approx 8(1.1-1) + 2 \approx 2.8$

$\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2) > 0$

since $\frac{d^2y}{dx^2} > 0$, the function must be concave up, so the line lies below the curve, so the approximation is an underestimation



c) Solve differential equation $\frac{dy}{dx} = xy^3$

$dy = xy^3 dx$

$\frac{dy}{y^3} = x\,dx$

$\int y^{-3} dy = \int x\,dx$

$\frac{y^{-2}}{-2} = \frac{x^2}{2} + C$

$-\frac{1}{2}y^{-2} = \frac{x^2}{2} + C$

solve for C $(1,2)$

$\frac{-1}{2(2)^2} = \frac{1^2}{2} + C$

$-\frac{1}{8} = \frac{1}{2} + C$

$\frac{-5}{8} = C$

$\frac{-1}{2y^2} = \frac{x^2}{2} - \frac{5}{8}$

$-2\left(\frac{-1}{2y^2} = \frac{x^2}{2} - \frac{5}{8}\right)$

$\frac{1}{y^2} = -x^2 + \frac{10}{8}$

$\frac{1}{y^2} = \frac{-x^2 + 5/4}{1}$

$y^2\left(-x^2 + \frac{5}{4}\right) = 1$

$y^2 = \frac{1}{-x^2 + \frac{5}{4}}$

$\boxed{y = \sqrt{\frac{1}{-x^2 + 5/4}}}$

or $f(x) = \frac{2}{\sqrt{5 - 4x^2}}$