

1a) What is a one-to-one function?

A function where each x-value has one unique y-value. It passes the horizontal line test and its inverse is a function.

b) What does monotonic mean?

A function that is always increasing or always decreasing

ex:  $y = x^3$

c. What is a local maximum?

a relative maximum.

2)

The radius of a sphere is increasing at a constant of 2 cm/sec. At the instant when the volume of the sphere is increasing at  $32\pi$  cm<sup>3</sup>/sec, the surface area of the sphere is

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \left(\frac{dr}{dt}\right)$

$32\pi = 4\pi r^2(2)$

Since  $S = 4\pi r^2$

$S = 4\pi r^2$

$\frac{dS}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$

$\frac{32\pi}{2} = 4\pi r^2$

and  $16\pi = 4\pi r^2$

$\frac{dr}{dt} = 2 \text{ cm/s}$

$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$

$16\pi = 4\pi r^2$

then  $S = 16\pi \text{ cm}^2$

$\frac{dV}{dt} = 32\pi \text{ cm}^3/\text{sec}$

$\frac{dV}{dt} = 4\pi r^2(2)$

3. A particle moves along the x-axis so that its velocity  $v$  at time  $t$ , for  $0 \leq t \leq 5$ , is given by

$v(t) = \ln(t^2 - 3t + 3)$

$\frac{d}{dx} \ln u = \frac{u'}{u}$

The particle is at the point  $x = 8$  at time  $t = 0$ .

$a(t) = \frac{2t-3}{t^2-3t+3}$

$a(4) = \frac{8-3}{16-12+3} = \frac{5}{7}$

(a) Find the acceleration of the particle at time  $t = 4$ .(b) Find all the times in the open interval  $0 < t < 5$  at which the particle changes direction.During which time intervals, for  $0 < t < 5$ , does the particle travel to the left?# Set  $v(t) = 0$  to first determine when particle is motionless.

$0 = \ln(t^2 - 3t + 3)$

$1 = t^2 - 3t + 3$



$0 = \log_e(t^2 - 3t + 3)$

$0 = t^2 - 3t + 2$

$e^0 = t^2 - 3t + 3$

$0 = (t-2)(t-1)$

Particle travels left (1, 2)

4.

$t=1, t=2$

b/c  $v(t) < 0$ 

A spherical balloon is inflated so that its volume is increasing at the rate of  $20$  cubic feet per minute. How fast is the surface area of the balloon increasing when the radius is  $4$  feet?

minute)

$V = \frac{4}{3}\pi r^3$

$r = 4$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$\frac{20}{4\pi(16)} = \frac{dr}{dt}$

$\frac{5}{16\pi} = \frac{dr}{dt}$

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi(4) \left(\frac{5}{16\pi}\right)$

$\frac{dS}{dt} = 10 \text{ ft}^2/\text{min}$

Find  $\frac{dS}{dt} =$  \_\_\_\_\_

$20 = 4\pi(4)^2 \frac{dr}{dt}$

5.

How many points of inflection does the graph of  $y = \frac{\sin x}{x}$  have on the interval  $(-\pi, \pi)$ ?  
OMIT

6.

$$f(x) = x^2 \quad g(x) = -x^{1/2}$$

At what value(s) of  $x$  do the graphs of  $y = x^2$  and  $y = -\sqrt{x}$  have perpendicular tangent lines?

\* Since perpendicular slopes are opposite reciprocals of each other, we can set the derivative of first function equal to the opposite reciprocal of 2nd functions' derivative.

$$f'(x) = 2x$$

$$g'(x) = -\frac{1}{2}x^{-1/2} = -\frac{1}{2\sqrt{x}} \rightarrow \text{opposite reciprocal} \rightarrow \frac{2\sqrt{x}}{1}$$

- 7) A spherical balloon is being inflated. At the instant when the rate of increase of the volume of the sphere is four times the rate of increase of the radius, the radius of the sphere is

$$2x = 2\sqrt{x}$$

$$(2x)^2 = (2\sqrt{x})^2$$

$$4x^2 = 4x$$

$$4x^2 - 4x = 0$$

$$4x(x-1) = 0$$

$$x=0, x=1$$

$V = \frac{4}{3}\pi r^3$	$\frac{dV}{dt} = \frac{4}{3} \cdot 3\pi r^2 \left(\frac{dr}{dt}\right)$	$4\left(\frac{dr}{dt}\right) = 4\pi r^2 \left(\frac{dr}{dt}\right)$	$\frac{1}{\pi} = r^2$	$r = \sqrt{\frac{1}{\pi}}$
$\frac{dV}{dt} = 4\left(\frac{dr}{dt}\right)$	$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$	$\frac{4\left(\frac{dr}{dt}\right)}{4\left(\frac{dr}{dt}\right)\pi} = r^2$	$\pm\sqrt{\frac{1}{\pi}} = \sqrt{r^2}$	

8. A particle moves in a vertical line so that its coordinate at time  $t$  is

$$y(t) = t^3 - 12t + 3, t \geq 0.$$

- (a) Find the velocity and acceleration functions.

$$v(t) = 3t^2 - 12$$

$$a(t) = 6t$$

- (b) When is the particle moving upward, and when is it moving downward?

$$0 = 3(t^2 - 4) \quad | \begin{array}{c} - \\ + \end{array} \quad t = 2, -2$$

$$0 = 3(t+2)(t-2) \quad | \begin{array}{c} \text{up on } (2, \infty) \\ \text{down on } [0, 2) \end{array}$$

- (c) Find the displacement of the particle during the first 3s.

$$y(0) = 0 - 0 + 3 = 3 \quad > \quad 3 - 6 = 9$$

$$y(3) = 3^3 - 12(3) + 3 = -6$$

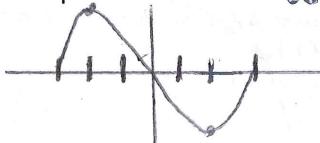
- (d) Find the distance that the particle travels in the time interval  $0 \leq t \leq 3$ .

$$y(0) = 3 > 16$$

$$y(2) = -13 > 7$$

$$y(3) = -6 > 7$$

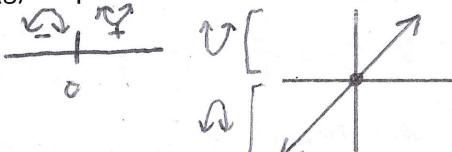
- (e) Given the position function below



- (f) Graph the velocity function



- (g) Graph the acceleration function



- (h) When is the particle speeding up? When is it slowing down?

$$a(t) = 6t \quad | \begin{array}{c} + \\ 0 \end{array} \quad v(t) \quad | \begin{array}{c} + \\ - \end{array} \quad | \begin{array}{c} + \\ + \end{array}$$

$$t=0$$

speeding up  $(2, \infty)$  b/c  $a(t) > 0$  and  $v(t) > 0$ , same signs

slowing down  $(0, 2)$  b/c  $a(t) > 0$  and  $v(t) < 0$ , opposite signs.

$v(t) < 0$ , opposite signs.

1

The  $\lim_{b \rightarrow 0} \frac{\ln(2x+b) - \ln(2x)}{b}$  is

- (A)  $\frac{1}{x}$        $f(x) = \ln(2x)$

(B)  $\frac{1}{2x}$        $f'(x) = \frac{1}{2x} \cdot 2 = \frac{2}{2x} = \boxed{\frac{1}{x}}$

(C)  $\ln x$   
 (D)  $\ln(2x)$   
 (E) Undefined

3

If  $f(3) = 8$ , then the slope of the tangent line to  $(f^{-1})'(8)$  must be

- |                       |             |                              |
|-----------------------|-------------|------------------------------|
| (A) $\frac{1}{f'(3)}$ | $f(3) = 8$  | $(f^{-1})(8) = 3$            |
| (B) $\frac{1}{f'(8)}$ |             |                              |
| (C) $f'(3)$           | $f'(3) = m$ | $(f^{-1})'(8) = \frac{1}{m}$ |
| (D) 3                 |             |                              |
| (E) 8                 |             |                              |

$$\frac{1}{f'(3)}$$

5 Find the inflection point(s) for the function  $f(x) = 2x(x + 4)^3$ .

- $$\begin{array}{l}
 \text{(A) } (0, 0) \\
 \text{(B) } (-4, 0), (-2, -32) \\
 \text{(C) } (0, 0), (-4, 0) \\
 \text{(D) } (-4, 0), (0, 0), (4, 0) \\
 \text{(E) } (-4, 8)
 \end{array}
 \quad
 \begin{aligned}
 f''(x) &= 2(x+4)(8x+8) + (x+4)^2/8 \\
 &= (x+4)[16x+16+8x+32] \\
 0 &= (x+4)(24x+48) \\
 x &= -4, 2 \quad \frac{+1}{-4} \quad \frac{-1}{-2} \quad \frac{+}{+}
 \end{aligned}$$

7

If the line tangent to  $y = f(x)$  at point  $(-3, 8)$  passes through the point  $(-2, 5)$  then,

- (A)  $f'(-2) = 3$   
 (B)  $f'(-2) = -3$   
 (C)  $f'(-3) = \frac{3}{5}$   
 (D)  $f'(-3) = -3$   
 (E)  $f(-2) = 5$

$$\text{slope of tangent line} = \frac{8-5}{-3-2} = \frac{3}{-1} = -3$$

Since slope of tangent line to curve at  $x = -3$  is  $-3$ , then  $f'(-3) = -3$

2

$$\text{What is } \lim_{h \rightarrow 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}?$$

- (A) 0       $f(x) = \tan x$   
 (B)  $\frac{\sqrt{2}}{2}$        $f'(x) = \sec^2 x$   
 (C) 1  
 (D)  $\frac{\sqrt{3}}{2}$        $f'(\frac{\pi}{4}) = [\sec(\frac{\pi}{4})]^2$   
 (E)  $\frac{2}{2}$        $= \frac{4}{2} = [ ]$

Let  $f$  be the function given by  $f(x) = \frac{x}{x-3}$ . For what value(s) of  $x$  is the slope of the line tangent to the graph of  $f$  at  $(x, f(x))$  equal to  $-\frac{3}{4}$ ?

- $$\begin{array}{ll}
 \text{(A)} \quad x = -\frac{3}{7} & f'(x) = \frac{1(x-3) - x(1)}{(x-3)^2} = \frac{x-3-x}{(x-3)^2} = \frac{-3}{(x-3)^2} \\
 \text{(B)} \quad x = \frac{9}{7} & \\
 \text{(C)} \quad x = 1 \text{ or } x = 5 & \left| \begin{array}{l} (x-3)^2 = 4 \\ x^2 - 6x + 9 = 4 \\ x^2 - 6x + 5 = 0 \\ (x-5)(x-1) = 0 \\ \boxed{x=1, 5} \end{array} \right. \\
 \text{(D)} \quad x = \pm 7 & \\
 \text{(E)} \quad \text{No solution} &
 \end{array}$$

6 At what value(s) of  $x$  do the graphs of  $y = e^x$  and  $y = x^2 + 5x$  have parallel tangent lines?

$y = e^x$ $y' = 2x + 5$	$e^x = 2x + 5$ $e^x - 2x - 5 = 0$	 $x = -2.457, x = 2.252$
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8

Let  $f(x) = x^3 - x$  such that  $f$  is continuous on a closed interval  $[-1, 1]$ . Find the critical number(s),  $c$ , that satisfies the mean value theorem for the given function and interval.

- $$(A) \pm \sqrt{\frac{1}{3}} \quad \text{set } f'(c) = \frac{f(b)-f(a)}{b-a}$$

- $$\begin{array}{ll} \text{(B)} & \pm\sqrt{3} \quad f(-1) = 0 \\ \text{(C)} & 0 \quad f(1) = 0 \quad \text{so} \quad \frac{f(-1)-f(1)}{-1-1} = \frac{0-0}{-2} = 0 \\ \text{(D)} & \sqrt[3]{\frac{1}{3}} \quad f'(x) = 3x^2 - 1 \end{array}$$

$$(E) \quad \frac{1}{3} \quad \text{set } 3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm \sqrt{\frac{1}{3}}$$

$$x = \pm \frac{1}{\sqrt{3}}$$

9.

$$f(-3) = \sqrt{3^2 + 16} = \sqrt{25} = 5$$

The local linear approximation for  $f(x) = \sqrt{x^2 + 16}$  near  $x = -3$  is

- (A)  $5 - \frac{3}{5}(x - 3)$    (B)  $5 + \frac{3}{5}(x - 3)$    (C)  $5 - \frac{3}{5}(x + 3)$   
 (D)  $3 - \frac{5}{3}(x - 3)$    (E)  $3 + \frac{3}{5}(x + 3)$

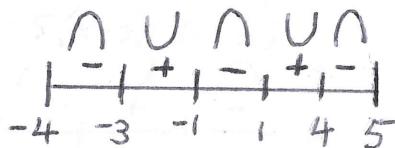
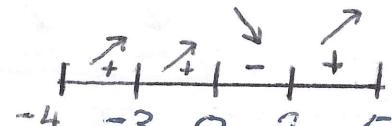
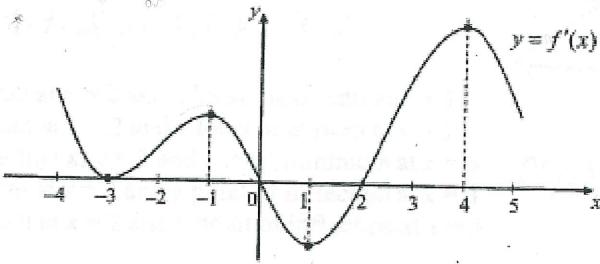
point:  $(-3, 5)$

$$f(x) = \frac{1}{2}(x^2 + 16)^{-1/2} (2x) \\ = \frac{x}{\sqrt{x^2 + 16}}$$

$$\begin{aligned} f'(-3) &= \frac{-3}{\sqrt{9+16}} = \frac{-3}{5} \quad m = \frac{-3}{5} \\ y - y_1 &= m(x - x_1) \\ y - 5 &= \frac{-3}{5}(x + 3) \\ y &= 5 - \frac{3}{5}(x + 3) \end{aligned}$$

11.

pos. slope  
slope = 0  
neg. slope



The figure above shows the graph of  $f'$ , the derivative of the function  $f$ , for  $-4 \leq x \leq 5$ . The graph of  $f'$  has horizontal tangent lines at  $x = -3, -1, 1$ , and  $4$ .

- (a) Find all values of  $x$ , for  $-4 < x < 5$ , for which  $f$  is decreasing. Justify your answer.

$$(0, 2) \text{ b/c } f'(x) < 0$$

- (b) Find all values of  $x$ , for  $-4 < x < 5$ , at which  $f$  attains a relative maximum. Justify your answer.

Rel. max at  $x = 0$  b/c  $f'(x)$  changes from + to -

- (c) Find all values of  $x$ , for  $-4 < x < 5$ , for which the graph of  $f$  is concave up.

$$(-3, -1), (1, 4) \text{ b/c } f''(x) > 0$$

- 12 A particle moves along the  $y$ -axis so that its position at any time  $t \geq 0$  is given by  $y(t) = t^2 - 4 \ln(t+1) - 1$ .

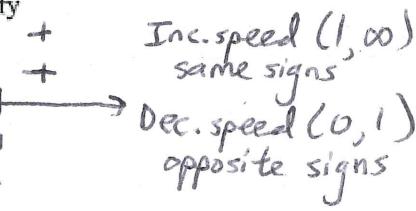
- (a) Find the velocity  $v(t)$  at any time  $t \geq 0$ .

$$v(t) = 2t - \frac{4}{t+1} = \frac{2t(t+1) - 4}{t+1}$$

- Find all values of  $t$  for which the speed of the particle is increasing. Justify your answer.

$$a(t) = 2 + 4(t+1)^{-2} = 2 + \frac{4}{(t+1)^2}$$

$$\frac{2t^2 + 2t - 4}{t+1} = \frac{2(t^2 + t - 2)}{t+1} = \frac{2(t+2)(t-1)}{t+1} \quad t = -1, 2, 1$$



- c. Find the total distance traveled by the particle from time  $t = 0$  to time  $t = 2$ .

$$y(0) = -1 > 1.773$$

$$y(1) = -2.773 > 1.379$$

$$y(2) = -1.394$$

10

At time  $t \geq 0$ , the position of a particle moving along the  $x$ -axis is given by

$x(t) = \frac{t^3}{3} + 2t + 2$ . For what value of  $t$  in the interval  $[0, 3]$  will the instantaneous velocity of the particle equal the average velocity of the particle from time  $t = 0$  to time  $t = 3$ ?

- (A) 1   (B)  $\sqrt{3}$    (C)  $\sqrt{7}$    (D) 3   (E) 5

$$\text{MVT} \quad x'(t) = \frac{1}{3} \cdot 3t^2 + 2$$

$$x(3) = 17 \quad \text{set } t^2 + 2 = 5$$

$$x(0) = 2$$

$$\frac{x(3) - x(0)}{3-0} = \frac{17-2}{3} = \frac{15}{3} = 5$$

$$t^2 = 3 \quad t = \sqrt{3}, -\sqrt{3}$$

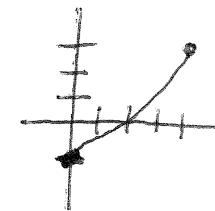
- 1) If  $f$  is a differentiable function and  $f(0) = -1$  and  $f(4) = 3$ , then which of the following must be true?

~~IVT~~ I. There exists a  $c$  in  $[0, 4]$  where  $f(c) = 0$ .  $\frac{f(4) - f(0)}{4 - 0} = \frac{3 - (-1)}{4 - 0} = \frac{4}{4} = 1$

~~Rolle's~~ fails II. There exists a  $c$  in  $[0, 4]$  where  $f'(c) = 0$ .

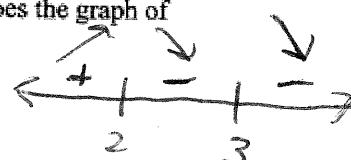
MVT III. There exists a  $c$  in  $(0, 4)$  where  $f'(c) = 1$ .

- (A) I only      (B) II only      (C) I and II only  
 (D) I and III only      (E) I, II, and III

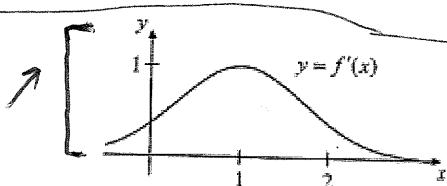
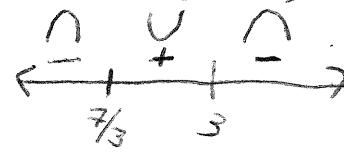


- 2) If  $f'(x) = -5(x-3)^2(x-2)$ , which of the following features does the graph of  $f(x)$  have? *Critical points: x = 3, 2*

- (A) a local minimum at  $x = 2$  and a local maximum at  $x = 3$   
 (B) a local maximum at  $x = 2$  and a local minimum at  $x = 3$   
 (C) a point of inflection at  $x = 2$  and a local minimum at  $x = 3$   
 (D) a local minimum at  $x = 2$  and a point of inflection at  $x = 3$   
 (E) a local maximum at  $x = 2$  and a point of inflection at  $x = 3$

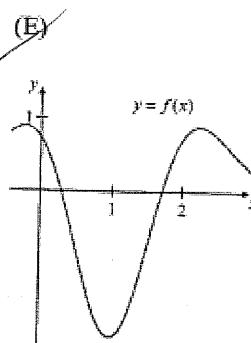
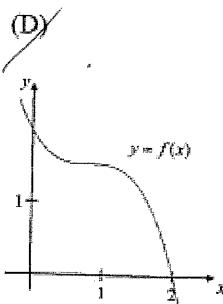
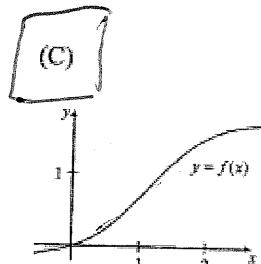
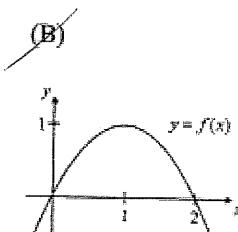
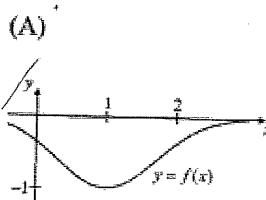


$$f''(x) = -5(x-3)(3x-7)$$



- Above x-axis, always positive slope
- POI at  $x = 1$
- graph is steepest at  $x = 1$

- 3) The graph of  $f'(x)$  is shown above. Which of the following could be the graph of  $f(x)$ ?



- 4) The radius of a sphere increases at a constant rate of 2 cm/min. At the time when the volume of the sphere is  $40 \text{ cm}^3$ , what is the rate of increase of the volume in  $\text{cm}^3/\text{min}$ ?  $V = \frac{4}{3}\pi r^3$

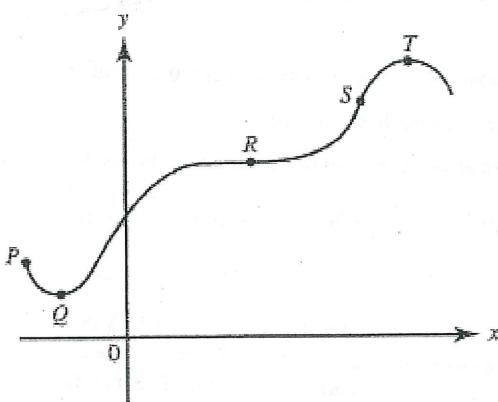
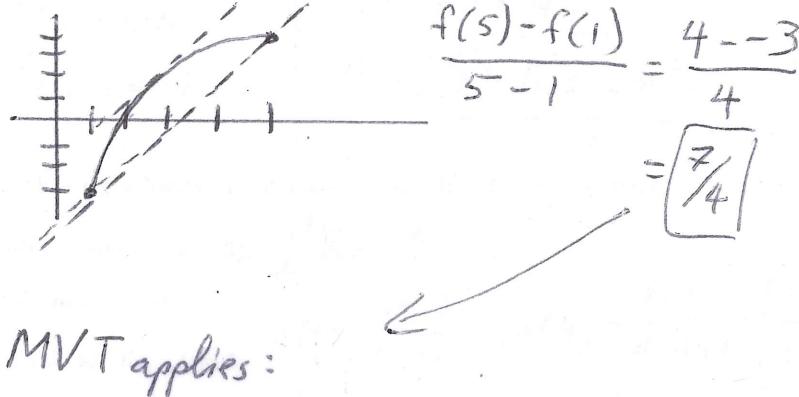
$$\frac{dr}{dt} = 2 \text{ cm/min}, V = 40, \text{ Find } \frac{dV}{dt}$$

- (A) 2.122 (B) 9.549 (C) 56.562 (D) 113.124 (E) 293.954

$$V = \frac{4}{3}\pi r^3 \quad \left| \begin{array}{l} 40 \cdot \frac{3}{4\pi} = r^3 \\ \sqrt[3]{30/\pi} = r \end{array} \right. \quad \left| \begin{array}{l} \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \left( \frac{dr}{dt} \right) \\ = 4(\pi)(2.122)^2(2) \end{array} \right. \quad \left| \begin{array}{l} \frac{dV}{dt} = 113.124 \text{ cm}^3/\text{min} \end{array} \right.$$

- 5)  $f(x)$  is a differentiable function with  $f(1) = -3$  and  $f(5) = 4$ . Which of the following must be true?

- (A)  $f(0) = k$  for some  $k$  in  $(1, 5)$   
 (B)  $f(x)$  is increasing on  $(1, 5)$   
 (C)  $f'(x) = \frac{7}{4}$  for all  $x$  in  $(1, 5)$   
 (D)  $f'(k) = 0$  for some  $k$  in  $(1, 5)$   
 (E)  $f'(k) = \frac{7}{4}$  for some  $k$  in  $(1, 5)$



6) slope = 0 concavity = 0

At which labeled point do both  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  equal zero?

- (A) P (B) Q (C) R (D) S (E) T

7) slope > 0 concavity = 0  
At which labeled point is  $\frac{dy}{dx}$  positive and  $\frac{d^2y}{dx^2}$  equal to zero?

- (A) P (B) Q (C) R (D) S (E) T

8) concave down  
At which labeled point is  $\frac{dy}{dx}$  equal to zero and  $\frac{d^2y}{dx^2}$  negative?

- (A) P (B) Q (C) R (D) S (E) T

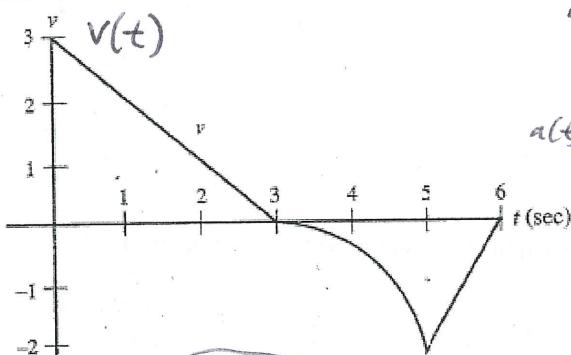
- 9) For how many values of  $t$  in the interval  $0 < t < 6$  is the acceleration undefined?

- (A) none (B) one (C) two (D) three (E) four

acceleration is slope of velocity  
a(t) undefined at  $t=3, t=5$

- 10) During what time interval (in sec) is the speed increasing?

- (A)  $0 < t < 3$  (B)  $3 < t < 5$  (C)  $3 < t < 6$   
 (D)  $5 < t < 6$  (E) never
- $v(t)$
- where  $v(t)$  and  $a(t)$  have same signs.



Find Avg. slope

$$v(5) = -1 \quad v(0) = 3$$

$$\text{avg. acceleration} = \frac{v(5) - v(0)}{5 - 0}$$

$$= \frac{-1 - 3}{5 - 0} = \frac{-4}{5} = \boxed{-\frac{4}{5}}$$

- 11) What is the average acceleration (in units/sec<sup>2</sup>) during the first 5 seconds?

- (A)  $-\frac{5}{2}$  (B)  $-1$  (C)  $-\frac{1}{5}$  (D)  $\frac{1}{5}$  (E)  $\frac{1}{2}$