$\boxed{\text{Key}}$

AP FRQ Review: Riemann Sums

(Calculator Active)

Question 1

4 5 6 5

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

smooth curve w(t)

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

a) Approximate $w'(12)$ using slope: $w'(12) \approx \frac{\text{change in temp.}}{\text{change in time}} \rightarrow \frac{67.9-61.8}{15-9}$

*water temperature is increasing at rate of 1.107 °F/min at t=12 mins.

$= \boxed{1.107\ °F/min.}$

b) $\int_0^{20} w'(t)dt = w(20) - w(0)$ *Recall FFTC: $\int_a^b f'(x)dx = f(b) - f(a)$

$w(20) - w(0) = 71.0 - 55.0 = \boxed{16}$ *The temperature increased 16°F over the course between t=0 to t=20 mins.

c) $\frac{1}{20}\int_0^{20} w(t)dt = \frac{1}{20}\left[4(55) + 5(57.1) + 6(61.8) + 5(67.9)\right] = \boxed{60.79°F}$

This estimation underapproximates since we are using left-Riemann Sum and increasing graph

d) *recall: final value = initial value + displacement

$w(b) = w(a) + \int_a^b w'(t)dt$

$w(25) = w(20) + \int_{20}^{25} 0.4\sqrt{t}\cos(0.06t)dt$

$= 71.0 + 2.043155$

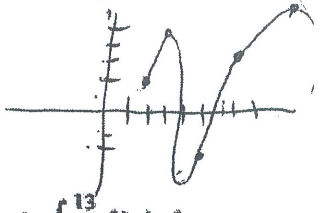
$\boxed{w(25) = 73.043°F}$

2. Riemann Sums

1 2 3 5

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.



(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

concave down

$= \frac{-2-4}{5-3} = \frac{-6}{2} = -3$

a) $f'(4) \approx \frac{f(5)-f(3)}{5-3} = -3$

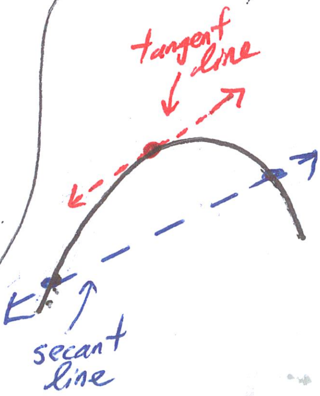
FFTC $\int_a^b f'(x)dx = f(b)-f(a)$

b) $\int_2^{13} 3dx - 5\int_2^{13} f'(x)dx = 3x\Big]_2^{13} - 5[f(13)-f(2)]$

$= 39-6-5[6-1]$

$= 33-25 = \boxed{8}$

c) $\int_2^{13} f(x)dx \approx 1(1)+2(4)+3(-2)+5(3) = \boxed{18}$



d) $f'(5)=3$ at point $(5,-2)$

Tangent line: $y+2=3(x-5)$ $y=3x-15-2$ $y=3x-17$

$y(7) \approx 3(7)-17 = 4$. Since graph is concave down for $5 \le x < 8$, tangent line will overapproximate value of $f(x)$. $f(7) \le 4$

secant line: use $(5,-2)$ and $(8,3)$

slope: $\frac{3--2}{8-5} = \frac{5}{3}$ $y+2 = \frac{5}{3}(x-5)$

$y = \frac{5}{3}(x-5)-2$

$y(7) \approx \frac{5}{3}(7-5)-2 = 4/3$. | secant line lies below the graph, therefore $y(7) > 4/3$.

$\frac{10}{3} - \frac{6}{3} =$

3) (Non-Calculator)

12 8 4 16

Question 3

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem.

Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

a) $v'(16) \approx \frac{240-200}{20-12} = \frac{40}{8} = 5 \text{ meters/min}^2$

approximate rate of change by finding slope at 2 points.

b) $\int_0^{40} |v(t)|\,dt$ is total distance Johanna jogs in interval $0 \le t \le 40$ mins.

$$\int_0^{40} |v(t)|\,dt \approx 12(200) + 8(240) + 4(220) + 16(150) = \boxed{7600 \text{ meters}}$$

convert to positive 220

c) velocity: $B(t) = t^3 - 6t^2 + 300$ | $B'(5) = 3(5)^2 - 12(5) = \boxed{15 \text{ meters/min}^2}$

acceleration: $B'(t) = 3t^2 - 12t$

d) Avg. velocity $= \frac{1}{b-a}\int_a^b v(t)\,dt \rightarrow \frac{1}{10-0}\int_0^{10} t^3 - 6t^2 + 300\,dt$

$$\frac{1}{10} \cdot \frac{t^4}{4} - \frac{6t^3}{3} + 300t \Big]_0^{10} = \boxed{\frac{1}{10}\left(\frac{10^4}{4} - \frac{6(10)^3}{3} + 300(10) - 0\right) \text{meters/min}}$$

$= 350$ meters/min.

1 2 2 3

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

AB/BC #4

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

Related Rates

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$\frac{dx}{dt} = 0.03$

a) $H'(6) \approx \frac{H(7)-H(5)}{7-5} = \frac{11-6}{7-2} = \boxed{\frac{5}{2} \text{ meters/yr.}}$

MVT: $f'(c) = \frac{f(b)-f(a)}{b-a}$

b) By Mean Value Theorem, since $\frac{H(5)-H(3)}{5-3} = \frac{6-2}{5-3} = \frac{4}{2} = 2$, there must be at least one time t, $2<t<10$ where $H'(t)=2$

trapezoid Area $\frac{w}{2}[h_1+h_2]$

c) Avg. height → *use Avg. value theorem: $\frac{1}{b-a}\int_a^b H(t)dt$

$$\text{Avg. height} = \frac{1}{10-2}\int_2^{10} H(t)dt = \frac{1}{8}\left[\frac{1}{2}[1.5+2] + \frac{2}{2}[2+6] + \frac{2}{2}[6+11] + \frac{3}{2}[11+15]\right]$$

$$= \boxed{\frac{263}{32}}$$

d) height is G(x)

Find $\frac{dG}{dt}$

$G = \frac{100x}{1+x}$

$\frac{dx}{dt} = 0.03$ meters/yr.

*Related Rates *quotient rule

$$\frac{dG}{dt} = \frac{100(\frac{dx}{dt})(1+x) - 100x(\frac{dx}{dt})}{(1+x)^2}$$

$$\frac{dG}{dt} = \frac{100(0.03)(1+1) - 100(1)(0.03)}{(1+1)^2}$$

$$= \frac{3}{4} \text{ meters/yr.}$$

*when $G = 50$,

$G = \frac{100x}{1+x}$

$50 = \frac{100x}{1+x}$

$50 + 50x = 100x$

$50 = 50x$ $\boxed{x=1}$