Key

**AP FRQ Review: Limits/Continuity**

1) Non-Calculator

Let $f$ be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that $f$ is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of $x$ for which $f'(x) = -3$.

(c) Find the average value of $f$ on the interval $[-1, 1]$.

a) *Step through continuity conditions:

i) $f(c)$ exists

ii) $\lim_{x\to c} f(x)$ exists $\left(\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\right)$

iii) $f(c) = \lim_{x\to c} f(x)$

✓ i) $f(0) = 1 - 2\sin(0) = 1$

✓ ii) $\lim_{x\to 0^-} 1 - 2\sin x = 1$ $\quad \lim_{x\to 0^+} e^{-4x} = e^0 = 1$

Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$, then $\lim_{x\to 0} f(x) = 1$

✓ iii) $f(0) = \lim_{x\to 0} f(x)$, therefore $f(x)$ is continuous at $x = 0$.

b) $f'(x) = \begin{cases} -2\cos x \\ e^{-4x} \cdot -4 \end{cases} \rightarrow f'(x) = \begin{cases} -2\cos x, & x < 0 \\ -4e^{-4x}, & x > 0 \end{cases}$

* test which piecewise function can be set equal to where $f'(x) = -3$

$-2\cos x \ne 3$ (for $x < 0$)

$-4e^{-4x} = -3$

$e^{-4x} = \frac{3}{4}$

$\ln e^{-4x} = \ln\left(\frac{3}{4}\right)$

$-4x = \ln\left(\frac{3}{4}\right)$

$\boxed{x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)}$

c) *Apply Avg. Value Theorem: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \rightarrow f(c) = \frac{1}{1-(-1)}\int_{-1}^{1} f(x)\,dx$

* $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1 - 2\sin x\,dx + \int_0^1 e^{-4x}\,dx$

u-sub
$u = -4x$
$\frac{du}{dx} = -4$
$dx = \frac{du}{-4}$

$\int e^u \cdot \frac{du}{-4}$

$-\frac{1}{4}\int e^u = -\frac{1}{4}e^u \rightarrow -\frac{1}{4}e^{-4x}\Big]_0^1$

$x - (-2\cos x)\Big]_{-1}^{0}$

$= 0 + 2\cos(0) - (-1 + 2\cos(-1)) + -\frac{1}{4}e^{-4} - \left(-\frac{1}{4}e^0\right)$

$= 2 + 1 - 2\cos(-1) - \frac{1}{4e^4} + \frac{1}{4} = \frac{13}{4} - 2\cos(-1) - \frac{1}{4e^4}$

Avg. value $= \frac{1}{2}\left[\frac{13}{4} - 2\cos(-1) - \frac{1}{4e^4}\right]$

$= \boxed{\frac{13}{8} - \cos(-1) - \frac{1}{8e^4}}$

2) Non-Calculator

The function $f$ is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of $f$ at $x = -3$.

(c) Let $g$ be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is $g$ continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

Apply chain rule

a) $f(x) = \sqrt{25-x^2} = (25-x^2)^{1/2}$

$f'(x) = \frac{1}{2}(25-x^2)^{-1/2}(-2x)$

$\boxed{f'(x) = \frac{-x}{\sqrt{25-x^2}}}$

b) $f'(-3) = \frac{-(-3)}{\sqrt{25-(-3)^2}} = \frac{3}{\sqrt{16}} = \frac{3}{4}$

$f(-3) = \sqrt{25-(-3)^2} = \sqrt{25-9} = 4$

point: $(-3, 4)$

slope: $m = 3/4$

$\boxed{y - 4 = \frac{3}{4}(x+3)}$

c) *Apply continuity conditions:

i) point exists: $f(c)$ exists

ii) $\lim_{x\to c} f(x)$ exists : $\left(\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\right)$

iii) $f(c) = \lim_{x\to c} f(x)$

i) $g(-3) = \sqrt{25-(-3)^2} = \sqrt{16} = 4$

ii) $\lim_{x\to -3^-} \sqrt{25-x^2} = 4$ $\quad \lim_{x\to -3^+} x+7 = -3+7 = 4$. Therefore, $\lim_{x\to -3} g(x) = 4$

iii) $g(-3) = \lim_{x\to -3} g(x) = 4$. Therefore, $g(x)$ is continuous at $x = -3$.

d) $\int_0^5 x(25-x^2)^{1/2} dx$ *Apply u-sub

$u = 25 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\int \cancel{x} \cdot u^{1/2} \cdot \frac{du}{-2\cancel{x}}$

$-\frac{1}{2}\int u^{1/2} du$

$-\frac{1}{2} \cdot \left(\frac{u^{3/2}}{3/2}\right) \to -\frac{1}{2}\cdot\frac{2}{3} u^{3/2} \to -\frac{1}{3}u^{3/2} \to -\frac{1}{3}(25-x^2)^{3/2}$

$-\frac{1}{3}(25-x^2)^{3/2}\Big]_0^5 = -\frac{1}{3}(25-5^2)^{3/2} - \left(-\frac{1}{3}(25-0)^{3/2}\right)$

$= 0 + \frac{1}{3}(25)^{3/2}$

$= \frac{1}{3}(125) = \boxed{\frac{125}{3}}$