Key

AP FRQ Review: "Curve Sketching"

1) Non-Calculator

Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function f.

a) * find critical points by setting $f'(x) = 0$, then create sign line, then test the intervals:

$f'(x) = (4-x)x^{-3}$

$0 = \dfrac{4-x}{x^3}$

critical points: $x = 4$, $x = 0$

$f'(x)$: $+$ ↗ at 1, $-$ ↘ at 5; number line 0, 4

$\boxed{\text{Relative max at } x = 4 \text{ since } f'(x) \text{ changes from } + \text{ to } -}$

b) * find $f''(x)$, set $f''(x) = 0$, find critical pts, sign line, test intervals:

$f'(x) = \underbrace{(4-x)}_{f}\underbrace{x^{-3}}_{g}$

$f''(x) = \underbrace{(-1)}_{f'} \cdot \underbrace{x^{-3}}_{g} + \underbrace{(4-x)}_{f} \cdot \underbrace{-3x^{-4}}_{g'}$

$f''(x) = \dfrac{-1}{x^3} - 12x^{-4} + 3x^{-3} \rightarrow \dfrac{-1}{x^3} - \dfrac{12}{x^4} + \dfrac{3}{x^3}$

$\rightarrow \dfrac{2}{x^3} - \dfrac{12}{x^4} \rightarrow \dfrac{2x}{x^4} - \dfrac{12}{x^4} \rightarrow f''(x) = \dfrac{2(x-6)}{x^4}$

critical points: $x = 6$, $x = 0$

$2(x-6) = 0$

$x^4 = 0$

$f''(x)$: $-$ at 1, $+$ at 7; number line 0, 6

$\boxed{f(x) \text{ is concave down on interval } 0 < x < 6 \text{ since } f''(x) < 0}$

c) final position = initial position + displacement

$f(x) = f(a) + \int_a^x f'(x)$

$f(x) = f(1) + \int_1^x 4t^{-3} - t^{-2}\,dt$

$\rightarrow \dfrac{4t^{-2}}{-2} - \dfrac{t^{-1}}{-1} \rightarrow \left.-\dfrac{2}{t^2} + \dfrac{1}{t}\right]_1^x$

$\dfrac{-2}{x^2} + \dfrac{1}{x} - \left(\dfrac{-2}{1^2} + \dfrac{1}{1}\right)$

$f(x) = 2 + \left(\dfrac{-2}{x^2} + \dfrac{1}{x} + 1\right)$

$\boxed{f(x) = 3 - \dfrac{2}{x^2} + \dfrac{1}{x}}$

2) Calculator

The function g is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

(a) Find all values of x in the interval $0.12 \le x \le 1$ at which the graph of g has a horizontal tangent line.

(b) On what subintervals of $(0.12, 1)$, if any, is the graph of g concave down? Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 0.3$.

(d) Does the line tangent to the graph of g at $x = 0.3$ lie above or below the graph of g for $0.3 < x < 1$? Why?

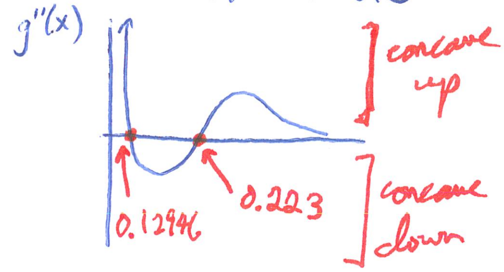
a) *horizontal tangents occur where $g'(x) = 0$ (Graph $g'(x)$ and look for x-intercepts)

$\boxed{x = 0.163 \text{ and } x = 0.359}$

b) *First, find critical points: set $g''(x) = 0$, create sign line, test intervals: (Graph $g''(x)$ in calculator and find x-intercepts: $x = 0.12946$ and $x = 0.223$

$g''(x)$: 0.12 | + ∪ | 0.12946 | − ∩ | 0.223 | + ∪ | 1

The graph of g is concave down on $(0.1294, 0.2227)$ since $g''(x) < 0$



c) slope: $g'(3) = \sin\left(3 + \frac{1}{3}\right) \approx -0.4722$

point: final position = initial position + displacement

$g(0.3) = g(1) + \int_1^{0.3} g'(x)$

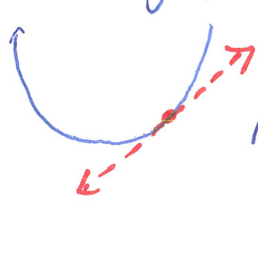
$g(0.3) = 2 + -0.0778 = 1.546$

point: $(0.3, 1.546)$

slope: $m = -0.4722$

$\boxed{y - 1.546 = -0.4722(x - 0.3)}$

d) Since $g''(x) > 0$ for $0.3 < x < 1$, the line tangent to the graph lies below the graph (graph is concave up)



*tangent line to the graph will sit below the curve if curve is concave up at that point.

3) Non-Calculator

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by

$f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x\to 0^+} f(x)$.

a) point: $f(e^2) = \dfrac{\ln e^2}{e^2} = \dfrac{2\cdot\ln e}{e^2} = \dfrac{2}{e^2}$ | slope: $f'(e^2) = \dfrac{1-\ln e^2}{(e^2)^2} = \dfrac{1-2\ln e}{e^4}$

tangent line equation: $y - y_1 = m(x - x_1)$ $\quad f'(e) = \dfrac{-1}{e^4}$

$$\boxed{y - \frac{2}{e^2} = \frac{-1}{e^4}(x - e^2)}$$

b) * find critical point, set $f'(x) = 0$, create sign line, test intervals

$f'(x) = \dfrac{1-\ln x}{x^2} \rightarrow 1 - \ln x = 0$, $-\ln x = -1$ | $\ln x = 1$, $e^{\ln x} = e^1$ | $x = e$

f': 0 — 1 ($+$, ↗) — e — e^2 ($-$, ↘)

$\boxed{\text{Relative max at } x = e \text{ since } f'(x) \text{ changes from } + \text{ to } -.}$

c) * find $f''(x) = 0$, plot critical point on sign line, test intervals

$f'(x) = \dfrac{\overbrace{1-\ln x}^{f}}{\underbrace{x^2}_{g}} \quad f''(x) = \dfrac{\overbrace{-\frac{1}{x}}^{f'}(\overbrace{x^2}^{g}) - (\overbrace{1-\ln x}^{f})(\overbrace{2x}^{g'})}{\underbrace{(x^2)^2}_{g^2}} = \dfrac{-x - 2x + 2x\ln x}{x^4}$

$f''(x) = \dfrac{-3x + 2x\ln x}{x^4} \rightarrow \dfrac{x(-3 + 2\ln x)}{x^4} \rightarrow \boxed{\dfrac{-3 + 2\ln x}{x^3}}$

find critical point:
$-3 + 2\ln x = 0$
$2\ln x = 3$
$\ln x = \frac{3}{2}$
$e^{\ln x} = e^{3/2}$
$x = e^{3/2}$

$f''(x)$: 0 — 1 ($-$, ∩) — $e^{3/2}$ — e^2 ($+$, ∪)

$\boxed{\text{POI at } x = e^{3/2} \text{ since } f''(x) \text{ change signs}}$

d) $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x} \rightarrow \dfrac{-}{+} \rightarrow \boxed{-\infty \text{ or does not exist}}$

4) Non-Calculator

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

a) $f(x) = kx^{1/2} - \ln x$

$f'(x) = \frac{1}{2}kx^{-1/2} - \frac{1}{x}$

$f'(x) = \frac{1}{2}kx^{-1/2} - x^{-1}$

$f''(x) = -\frac{1}{4}kx^{-3/2} + x^{-2}$

$$\boxed{\begin{aligned} f'(x) &= \frac{k}{2\sqrt{x}} - \frac{1}{x} \\ f''(x) &= -\frac{k}{4x^{3/2}} + \frac{1}{x^2} \end{aligned}}$$

b) * critical point where $f'(x) = 0$

$f'(x) = \frac{k}{2\sqrt{x}} - \frac{1}{x}$

$0 = \frac{k}{2\sqrt{x}} - \frac{1}{x}$ plug in $x = 1$

$0 = \frac{k}{2\sqrt{1}} - \frac{1}{1}$

$1 = \frac{k}{2}$

$\boxed{k = 2}$

* we can apply 2nd derivative test: If $f'(1) = 0$, then find $f''(1)$ to determine if point is relative max or min on $f(x)$.

$f''(1) = \frac{-2}{4(1)^{3/2}} + \frac{1}{1} = \frac{1}{2} > 0$

Since $f''(1) > 0$ and $f'(1) = 0$ then $f(1)$ must be relative minimum on $f(x)$

c) * If graph of $f(x)$ has POI on the x-axis, then we can set $f''(x) = 0$ and $f(x) = 0$

finds POI

finds x-intercept

$f''(x) = \frac{-k}{4x^{3/2}} + \frac{1}{x^2}$

$0 = \frac{-k}{4x^{3/2}} + \frac{1}{x^2}$

$\frac{k}{4x^{3/2}} = \frac{1}{x^2}$

$kx^2 = 4x^{3/2}$

$k = \frac{4x^{3/2}}{x^2} = \frac{4}{x^{1/2}} = \frac{4}{\sqrt{x}}$

$f(x) = k\sqrt{x} - \ln x$

$0 = k\sqrt{x} - \ln x$

$\ln x = k\sqrt{x}$

$\frac{\ln x}{\sqrt{x}} = k$

set equal and solve for x

$\frac{4}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}}$

$\cancel{\sqrt{x}}\ln x = 4\cancel{\sqrt{x}}$

$e^{\ln x} = e^4$

$x = e^4$

$k = \frac{4}{\sqrt{x}}$

$\downarrow$

$\boxed{k = \frac{4}{e^2}}$