AP FRQ Review WS: Implicit Differentiation/2nd Derivative

1) (Non-Calculator)

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

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a) *use $\frac{dy}{dx}$ to find slope: $\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{3(1)^2-(-1)} = \frac{1}{4}$

point: $(-1,1)$
slope: $m = 1/4$

$\boxed{y-1 = \frac{1}{4}(x+1)}$

b) * vertical tangents occur where denominator of $\frac{dy}{dx} = 0$

set $3y^2 - x = 0$

$x = 3y^2$

plug back into given equation: $y^3 - xy = 2$

$y^3 - (3y^2)y = 2 \rightarrow y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$y = -1$

Since $y = -1$, plug back into original equation and solve for x:

$y^3 - xy = 2$

$(-1)^3 - x(-1) = 2$

$-1 + x = 2$

$x = 3$

The tangent line to curve is vertical at point $(3,-1)$

c) *Apply quotient rule to find $\frac{d^2y}{dx^2}$ from $\frac{dy}{dx} = \frac{y}{3y^2-x}$

$$\frac{d^2y}{dx^2} = \frac{\underbrace{1\left(\frac{dy}{dx}\right)}_{f'}\underbrace{(3y^2-x)}_{g} - \underbrace{(y)}_{f}\underbrace{\left(6y\frac{dy}{dx}-1\right)}_{g'}}{\underbrace{(3y^2-x)^2}_{g^2}}$$

*plug in $(-1,1)$ to $\frac{d^2y}{dx^2}$

*replace $\frac{dy}{dx}$ with $\frac{1}{4}$ since $\left.\frac{dy}{dx}\right|_{(-1,1)} = 1/4$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,1)} = \frac{(1)(1/4)(3(1)^2-(-1)) - 1(6(1)(1/4)-1)}{(3(1)^2-(-1))^2}$$

$$= \frac{1-\frac{1}{2}}{16} = \boxed{\frac{1}{32}}$$

2) (Non-Calculator)

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

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a) *Apply implicit differentiation and product rule

$$x^2 + 4y^2 = 7 + \underbrace{3x}_{f}\,\underbrace{y}_{g}$$

$$2x + 8y\left(\frac{dy}{dx}\right) = 0 + \underbrace{3}_{f'}\cdot\underbrace{y}_{g} + \underbrace{3x}_{f}\cdot\underbrace{1\left(\frac{dy}{dx}\right)}_{g'}$$

$$2x + 8y\left(\frac{dy}{dx}\right) = 3y + 3x\left(\frac{dy}{dx}\right)$$

$$8y\left(\frac{dy}{dx}\right) - 3x\left(\frac{dy}{dx}\right) = 3y - 2x$$

$$\frac{dy}{dx}\left(8y - 3x\right) = 3y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}}$$

b) *horizontal tangent occurs where numerator of $\frac{dy}{dx} = 0$

*plug in 3 for x

$3y - 2x = 0$

$3y - 3(2) = 0$

$3y - 6 = 0$

$3y = 6$

$y = 2$

*plug in (3, 2) to equation to confirm point is on the curve.

$x^2 + 4y^2 = 7 + 3xy$

$3^2 + 4(2)^2 = 7 + 3(3)(2)$

$9 + 16 = 7 + 18$

$25 = 25$ ✓

c) *Apply quotient rule to find $\frac{d^2y}{dx^2}$ using $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$

$$\frac{d^2y}{dx^2} = \frac{\underbrace{\left(3\left(\frac{dy}{dx}\right) - 2\right)}_{f'}\underbrace{\left(8y - 3x\right)}_{g} - \underbrace{\left(3y - 2x\right)}_{f}\underbrace{\left(8\frac{dy}{dx} - 3\right)}_{g'}}{\underbrace{\left(8y - 3x\right)^2}_{g^2}}$$

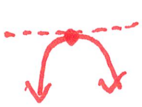
*plug in (3, 2) into $\frac{d^2y}{dx^2}$

*replace $\frac{dy}{dx} = 0$ since $\left.\frac{dy}{dx}\right|_{(3,2)} = 0$

$$\frac{d^2y}{dx^2} = \frac{(3(0) - 2)(8(2) - 3(3)) - \underbrace{\cancel{(3)(2) - 2(3)}}_{=0}(8(0) - 3)}{(8(2) - 3(3))^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-2}{7}}$$

By the 2nd derivative test, since $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$, Rel. max at point P.



Rel. max

If graph has slope = 0 and concave down, rel. max