Key

AP FRQ Review WS: Related Rates

1) Calculator

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

a) $r = 100$ cm

$h = 0.5$ cm

$\frac{dr}{dt} = 2.5$ cm/min

$\frac{dV}{dt} = 2000$ cm^3/min

Find $\frac{dh}{dt} = \ ?$

$V = \underbrace{\pi r^2}_{f} \underbrace{h}_{g}$

product Rule

$\frac{dV}{dt} = \underbrace{2\pi r\left(\frac{dr}{dt}\right)}_{f'} \underbrace{h}_{g} + \underbrace{\pi r^2}_{f} \underbrace{\left(\frac{dh}{dt}\right)}_{g'}$

$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$

$2000 = 785.398 + 31415.926 \frac{dh}{dt}$

$1214.602 = 31415.926\left(\frac{dh}{dt}\right)$

$\boxed{\frac{dh}{dt} = 0.039 \text{ cm/min}}$

b) * Rate of Change Volume (oil slick) = Rate of oil Added − Rate of oil removed

$\frac{dV}{dt} = 2000 - R(t)$

* Max volume occurs when $\frac{dV}{dt} = 0$

when $\frac{dV}{dt} = 0$,

$0 = 2000 - R(t)$

$R(t) = 2000$

$400\sqrt{t} = 2000$

$\sqrt{t} = 5 \rightarrow t = 25$

$\frac{dV}{dt}$: $+$ ↗ | 25 | $-$ ↘

Since $\frac{dV}{dt} > 0$ when $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, oil slick reaches max volume 25 mins after the removal device begins working.

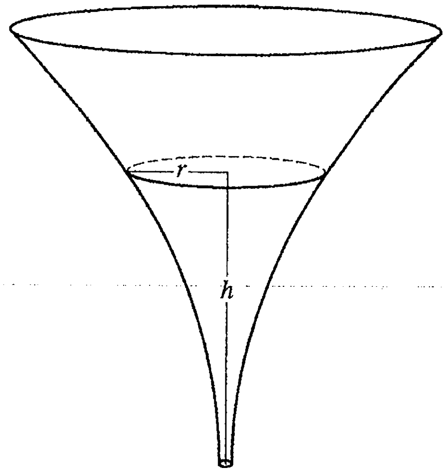
c) * Total Amount = Initial Amount + Amount Added − Amount Removed

$\boxed{A(25) = 60{,}000 + \int_0^{25} 2000\,dt - \int_0^{25} R(t)\,dt}$

Total Amount of oil at $t = 25$ min.

AB/BC

2)



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}(3 + h^2)$, where $0 \le h \le 10$. The units of r and h are inches.

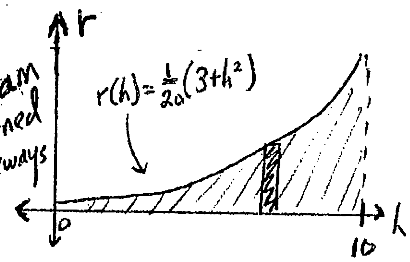
(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$\boxed{2}$ a) * Avg. value theorem $\frac{1}{b-a}\int_a^b r(h)\,dh = \frac{1}{10}\int_0^{10} \frac{1}{20}(3+h^2)$

* Diagram turned sideways



$$\text{Avg. value} = \frac{1}{10-0}\int_0^{10} \frac{3}{20} + \frac{h^2}{20}\,dh = \frac{1}{10}\cdot \left.\frac{3h}{20} + \frac{h^3}{60}\right]_0^{10}$$

$$= \frac{1}{10}\cdot\left(\frac{3(10)}{20} + \frac{10^3}{60} - 0\right) = \frac{1}{10}\cdot\left(\frac{30}{20} + \frac{1000}{60}\right) = \frac{1090}{600} = \boxed{\frac{109}{60}}$$

$\boxed{4}$ b) * Disc Method

$$R(h) = \frac{1}{20}(3+h^2) - 0 = \frac{1}{20}(3+h^2) \qquad V = \pi\int_0^{10} R(h)^2\,dh = \pi\int_0^{10}\left[\frac{1}{20}(3+h^2)\right]^2 dh$$

$$V = \pi\int_0^{10} \frac{1}{400}(9+6h^2+h^4)\,dh = \left.\frac{9}{400}h + \frac{6}{400}\left(\frac{h^3}{3}\right) + \frac{1}{400}\left(\frac{h^5}{5}\right)\right]_0^{10}$$

$$\boxed{= \frac{9}{400}(10) + \frac{6}{400}\left(\frac{10^3}{3}\right) + \frac{10^5}{400(5)} - 0 = \frac{2209}{40}\pi \text{ or } 55.225\pi}$$

$\boxed{3}$ c) * Related Rates: $r = \frac{1}{20}(3+h^2)$

$r = \frac{3}{20} + \frac{1}{20}h^2$

$h = 3$ in.

$\frac{dr}{dt} = \frac{-1}{5}$ $\quad \frac{dh}{dt} = $ ___?

$\frac{dr}{dt} = 0 + \frac{1}{20}\cdot 2h\left(\frac{dh}{dt}\right)$

$\frac{-1}{5} = \frac{2}{20}(3)\left(\frac{dh}{dt}\right)$

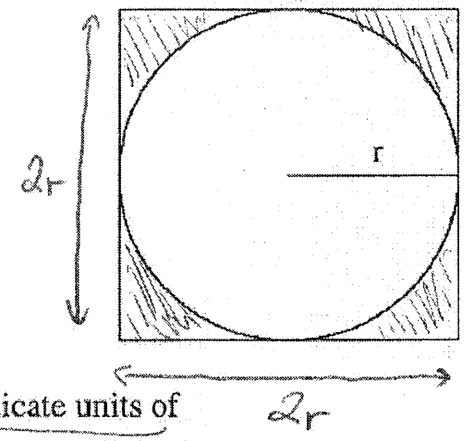
$\boxed{\frac{dh}{dt} = \frac{-2}{3} \text{ in/sec}}$

GO ON TO THE NEXT PAGE.

AP Calculus AB Related Rates WS Name Key Date

1994 AB5, BC2 3)

1 A circle is inscribed in a square as shown in the figure. The circumference of the circle is increasing at a constant rate of 6 inches per minute. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)



$\frac{dC}{dt} = 6$ in/min $\quad \frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$

a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

$P = 8r$

$\frac{dP}{dt} = 8\frac{dr}{dt}$

$6 = 2\pi\left(\frac{dr}{dt}\right)$

$\frac{6}{2\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi}$ in/min

$\frac{dP}{dt} = 8\left(\frac{3}{\pi}\right) = \boxed{\frac{24}{\pi} \text{ in/min.}}$

A_e = Area enclosed
A_c = Area circle
A_s = Area square

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

$A = 25\pi$

$\frac{dA_e}{dt} =$ ____

$A = \pi r^2$

$25\pi = \pi r^2$

$25 = r^2$

$\boxed{5 = r}$

$A_e = A_s - A_c$

$A_s = (2r)^2$

$A_c = \pi r^2$

$A_e = 4r^2 - \pi r^2$

$\frac{dA_e}{dt} = 8r\left(\frac{dr}{dt}\right) - 2\pi r\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \frac{3}{\pi}$ in/min

$\frac{dA}{dt} = 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$\boxed{\frac{dA}{dt} = \frac{120}{\pi} - 30 \text{ in}^2/\text{min}}$

$V = 4(t)^{1/2}$

2. Suppose that a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ in^3. How fast is the radius changing after 64 seconds? ($V = \frac{4}{3}\pi r^3$)

$\longrightarrow 32 = \frac{4}{3}\pi(r^3) \quad \frac{24}{\pi} = r^3 \quad r = \sqrt[3]{24/\pi}$

$t = 64$

$V = 4\sqrt{64} = 4 \cdot 8 = 32$ in^3

$\frac{dV}{dt} = 4 \cdot \frac{1}{2}t^{-1/2}\left(\cancel{\frac{dt}{dt}}\right)$

$\frac{dV}{dt} = \frac{2}{\sqrt{t}} = \frac{2}{\sqrt{64}} = \frac{2}{8} = \frac{1}{4}$ in^3/s

Find $\frac{dr}{dt} =$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

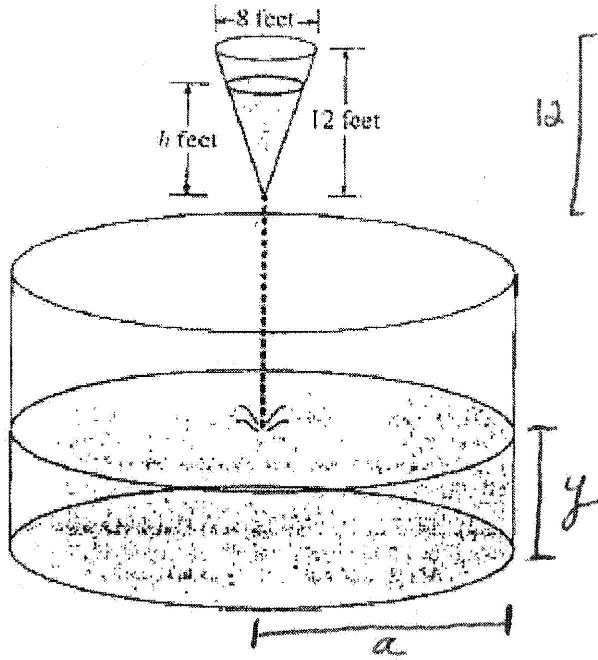
$\frac{1}{4} = 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2 \frac{dr}{dt}$

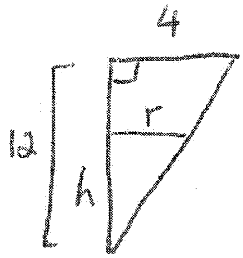
$\frac{1}{4 \cdot 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{194.974} \approx \boxed{0.005 \text{ in/s}}$

1995 AB 5

4)





$\frac{r}{4} = \frac{h}{12}$

$12r = 4h$

$r = \frac{4}{12}h$

$\boxed{r = \frac{h}{3}}$

$\frac{dh}{dt} = h - 12 \text{ ft/min}$

As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

$V = \frac{\pi}{3} r^2 h$

$V = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{h^2}{9} \cdot h$

$\boxed{V = \frac{\pi}{27} h^3}$

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

$V = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9} h^2 \cdot \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9} h^2 \cdot (h-12)$

$\frac{dV}{dt} = \frac{\pi}{9}(3)^2(3-12)$

$= \frac{\pi}{\cancel{9}} \cdot \cancel{9}(-9)$

$\boxed{\frac{dV}{dt} = -9\pi \text{ ft}^3/\text{min}}$

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure. $(V = \pi a^2 y)$

$\frac{dV}{dt} = 9\pi \text{ ft}^3/\text{min}$

Area(base) $= 400\pi$

$A = \pi r^2$

$400\pi = \pi r^2$

$400 = r^2$

$20 = r$

$r = 20 \text{ ft}$

$a = 20$

$V = \pi (20)^2 y$

$V = 400\pi y$

$\frac{dV}{dt} = 400\pi \left(\frac{dy}{dt}\right)$

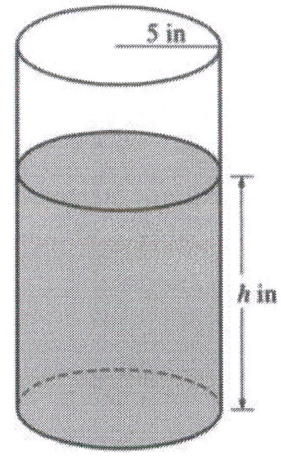
$9\pi = 400\pi \left(\frac{dy}{dt}\right)$

$\frac{9}{400} = \frac{dy}{dt}$

$\boxed{\frac{dy}{dt} = \frac{9}{400} \text{ ft/min}}$

5) (Non-Calculus)

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

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a) * Recall that even though cylinder has volume formula $(V=\pi r^2 h)$ that resembles a cone $(V=\frac{1}{3}\pi r^2 h)$, there is no similar triangle substitution needed, making cylinder problems easier.

* The radius for cylinder is a constant $(r=5)$

$V=\pi r^2 h$

$V=\pi(5)^2 h$

$V=25\pi h$

$\frac{dV}{dt}=25\pi\left(\frac{dh}{dt}\right)$

* Given: $\frac{dV}{dt}=-5\pi\sqrt{h}$

$-5\pi\sqrt{h}=25\pi\left(\frac{dh}{dt}\right)$

$\frac{-5\pi\sqrt{h}}{25\pi}=\frac{dh}{dt}$

$\boxed{\frac{dh}{dt}=\frac{-\sqrt{h}}{5}}$

b) * Solve differential equation: (Cross multiply, separate variables, take antiderivative)

$\frac{dh}{dt}=\frac{-\sqrt{h}}{5}$

$5dh=-\sqrt{h}\,dt$

$\frac{dh}{-\sqrt{h}}=\frac{dt}{5}$

$-h^{-1/2}dh=\frac{1}{5}dt$

$-\int h^{-1/2}dh=\frac{1}{5}\int 1dt$

$-\left(\frac{h^{1/2}}{1/2}\right)=\frac{1}{5}t+C$

$-2\sqrt{h}=\frac{1}{5}t+C$

find C: use (0,17) for (t,h)

$-2\sqrt{17}=\frac{1}{5}(0)+C$

$-2\sqrt{17}=C$

$-2\sqrt{h}=\frac{1}{5}t-2\sqrt{17}$

$\frac{-1}{2}\left(-2\sqrt{h}=\frac{1}{5}t-2\sqrt{17}\right)$

$(\sqrt{h})^2=\left(\frac{-1}{10}t+\sqrt{17}\right)^2$

$\boxed{h=\left(\frac{-1}{10}t+\sqrt{17}\right)^2}$

c) * coffeepot is empty when h(height) = 0.

$h=\left(\frac{-1}{10}t+\sqrt{17}\right)^2$

$\sqrt{0}=\sqrt{\left(\frac{-1}{10}t+\sqrt{17}\right)^2}$

$0=\frac{-1}{10}t+\sqrt{17}$

$\frac{1}{10}t=\sqrt{17}$

$\boxed{t=10\sqrt{17}}$

c) $\frac{dh}{dt} = \frac{-1}{10}\sqrt{h}$

$dh = \frac{-1}{10}\sqrt{h}\,dt$

$\frac{dh}{\sqrt{h}} = \frac{-1}{10}dt$

$h^{-1/2}dh = \frac{-1}{10}dt$

$\int h^{-1/2}dh = \frac{-1}{10}\int 1\,dt$

$\frac{h^{1/2}}{1/2} = \frac{-1}{10}t + C$

$2h^{1/2} = \frac{-1}{10}t + C$

$2(5)^{1/2} = \frac{-1}{10}(0) + C$ *solve for $h(0)$.

$2\sqrt{5} = C$

$2h^{1/2} = \frac{-1}{10}t + 2\sqrt{5}$

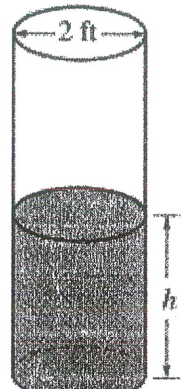
c)...(continued...)

$\frac{1}{2}\left(2h^{1/2} = \frac{-1}{10}t + 2\sqrt{5}\right)$

$h^{1/2} = \frac{-1}{20}t + \sqrt{5}$

$\left(h^{1/2}\right)^2 = \left(\frac{-1}{20}t + \sqrt{5}\right)^2$

$\boxed{h = \left(\frac{-1}{20}t + \sqrt{5}\right)^2}$



6) A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

2 (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

3 (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

4 (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

a) *Related Rates: Find $\frac{dV}{dt}$. $h = 4$ ft. radius (constant) = 1 ft.

*radius is constant

$V = \pi r^2 h$

$V = \pi(1)^2 h$

$V = \pi h$

$\frac{dV}{dt} = \pi\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \pi\left(\frac{-1}{10}\sqrt{h}\right)$

$\frac{dV}{dt} = \pi\left(\frac{-1}{10}\cdot\sqrt{4}\right)$

$\frac{dV}{dt} = \frac{-2\pi}{10} = \frac{-\pi}{5}\ \text{ft}^3/\text{sec}$

$\boxed{\frac{dV}{dt} = \frac{-\pi}{5}\ \text{ft}^3/\text{sec}}$

b) * rate of change increasing/decreasing refers to $\frac{d^2h}{dt^2}$ (derivative of $\frac{dh}{dt}$)

$\frac{dh}{dt} = \frac{-1}{10}h^{1/2}$

$\frac{d^2h}{dt^2} = \frac{-1}{10}\cdot\frac{1}{2}h^{-1/2}\left(\frac{dh}{dt}\right)$

$\frac{d^2h}{dt^2} = \frac{-1}{20\sqrt{h}}\left(\frac{-1}{10}\sqrt{h}\right)$

$\frac{d^2h}{dt^2} = \frac{1}{200} > 0$

Therefore, the rate of change of height of water is increasing since $\frac{d^2h}{dt^2} > 0$.

c) (see top of page...)