GEORGIA MATHEMATICS LEAGUE

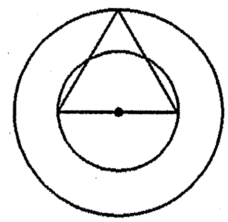
P.O. Box 12014, Columbus, Georgia 31917-2014

All official participants must take this contest at the same time.

Contest Number 2 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **November 10, 2015**

Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes NEXT CONTEST: DEC. 8, 2015 *Answer Column*

2-1. If 23 is written as the sum of the squares of 4 positive integers (not necessarily different), what is the largest square in this sum?	2-1.
2-2. One side of an equilateral triangle is the diameter of a circle, and one vertex of the triangle lies on a larger circle, concentric with the smaller circle, as shown. If the area of the smaller circle is 16π, what is the area of the larger circle? 	2-2.
2-3. When ten mathletes huddled together, they spaced themselves equally around a circle. The sum of the numbers on their uniforms was 300. If each number was the average of the two numbers nearest it, what was the largest of the ten numbers on their uniforms? 	2-3.
2-4. What are all real values of $x \neq 0$ that satisfy $\lvert x\rvert^{x^2-x-2} < 1$?	2-4.
2-5. What is the smallest integer $x > 1$ for which $\sqrt{x\sqrt{x\sqrt{x}}}$ is an integer?	2-5.
2-6. The length of each side of a triangle is the reciprocal of a different integer. If one of these integers is 2015, what is the least possible sum of the other two integers?	2-6.

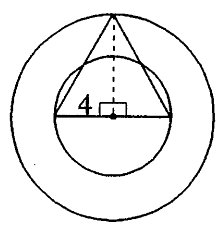
Problem 2-1

Since $23 = 9 + 9 + 4 + 1$, the largest square in this sum is $\boxed{9}$.

[By Lagrange's 4-Square Theorem, every positive integer can be written as the sum of the squares of 4 integers. This is the only way to write 23 as such a sum.]

Problem 2-2

The area of the smaller circle is 16π, so its radius is 4 and a side of the triangle is 8. The larger circle's radius, an altitude of the triangle, is $4\sqrt{3}$. The larger circle's area is $\pi(4\sqrt{3})^2 = \boxed{48\pi}$.



Problem 2-3

We shall show that all ten numbers are equal. If not, then look at the largest one, which is the average of its two nearest neighbors. If one neighbor were smaller, the other would have to be larger. This is impossible, so all three are equal. Similarly, all ten numbers must be equal, so each of the ten numbers is $\boxed{30}$.

Problem 2-4

Clearly, $|x| \neq 1$. To find solutions, consider 2 cases:

Case I: If $|x| > 1$, the exponent must be negative. Since $x^2-x-2 = (x-2)(x+1) < 0 \Leftrightarrow -1 < x < 2$, the solutions in the interval $|x| > 1$ are $\{x \mid 1 < x < 2\}$.

Case II: If $|x| < 1$, the exponent must be positive. Since $x^2-x-2 = (x-2)(x+1) > 0 \Leftrightarrow x < -1$ or $x > 2$, there are no solutions in the interval $|x| < 1$.

Only Case I works, so the solutions are $\boxed{1 < x < 2}$.

Problem 2-5

Method I: Square each side of $n = \sqrt{x\sqrt{x\sqrt{x}}}$ 3 times to get $n^8 = x^7$. If this has a solution in integers, n must be the 7th power of some integer and x must be the 8th power of an integer. The smallest such $x > 1$ must be the 8th power of 2, so $x = 2^8 = \boxed{256}$.

Method II: The innermost x will have its square root taken three times in succession, so the least $x > 1$ for which $\sqrt{x\sqrt{x\sqrt{x}}}$ is an integer must be a power of 2 for which the exponent can be divided by 2 three times in succession. That means that the exponent is $2^3 = 8$, so $x = 2^8 = 256$.

Problem 2-6

Call the integers a, b, 2015. To minimize $a+b$, make $a < b < 2015$. In any triangle, the longest side < the sum of the other sides, so $\frac{1}{a} < \frac{1}{b} + \frac{1}{2015} = \frac{2015+b}{2015b}$ $\Leftrightarrow a > \frac{2015b}{b+2015} \Leftrightarrow a+b > \frac{2015b}{b+2015} + b$. This inequality's right side is increasing as b increases. Its least value corresponds to b's least value. Since $b > a$, b's least value is $a+1$. The greater the sum on the right, the greater $\frac{1}{a}$ is ($\Leftrightarrow$ the less a is). Since $\frac{1}{a} < \frac{1}{(a+1)} + \frac{1}{2015}$, we get $a^2+a-2015 > 0$. Equality occurs when $a = \frac{-1+\sqrt{8061}}{2} = \frac{-1+89.783...}{2} > 44.39$. Thus, $a = 45$ and $a+b = 45+46 = \boxed{91}$.