GEORGIA MATHEMATICS LEAGUE

P.O. Box 12014, Columbus, Georgia 31917-2014

All official participants must take this contest at the same time.

Contest Number 5 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **February 9, 2016**

Name ____________________ **Teacher** __________ **Grade Level** ____ **Score** ___

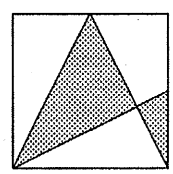
Time Limit: 30 minutes NEXT CONTEST: MAR. 15, 2016 *Answer Column*

5-1. If $-1 < x < 0$, then for what positive integer $n \le 2016$ does x^n take on its least value? — 5-1.

5-2. If we remove the first + sign and then "close" the resulting space in 1+2+3+4+5+6+7+8+9, we'll get 12+3+4+5+6+7+8+9 = 54. What is the ordinal number (first, second, etc.) of the + sign whose removal, followed by a "closing" of the resulting space, would make the resulting sum equal 99? — 5-2.

5-3. For how many different integers $b > 1$ is $\log_b 256$ a positive integer? — 5-3.

5-4. In the diagram, three segments are drawn interior to a square. Each segment connects a vertex of the square to the midpoint of a side of the square. If the area of the larger shaded triangle is 150, what is the area of the smaller shaded triangle? — 5-4.



5-5. Regular polygons M and N have m and n sides respectively, with $m > n$. What are all ordered pairs (m,n) for which the ratio of the measure of an interior angle of M to the measure of an interior angle of N is 3:2? — 5-5.

5-6. From a box containing 20 gold, 10 silver, and some bronze medals, the probability of randomly selecting 2 gold and 2 silver medals, with replacement, equals the probability of randomly selecting 1 gold, 1 silver, and 2 bronze medals, with replacement. How many bronze medals are in the box? — 5-6.

Problem 5-1

If $-1 < x < 0$ and n is even, then $x < 0 < x^n$. Similarly, if n is odd, then $-1 < x \le x^n < 0$, with equality if and only if $n = \boxed{1}$.

Problem 5-2

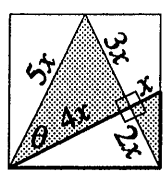
Start at the right and remove one + sign. We get an 89 (no), a 78 (no), a 67 (yes). Removing the sign between 6 and 7, we get $1+2+3+4+5+67+8+9 = 99$, as required. The $\boxed{\text{sixth}}$ + sign must be removed.

[Note: If we remove two plus signs, other solutions are $12+3+4+56+7+8+9$ and $1+23+45+6+7+8+9$.]

Problem 5-3

$\log_b 256 = n \Leftrightarrow b^n = 256$. Since $(2^8)^1 = (2^4)^2 = (2^2)^4 = (2^1)^8 = 256$, it follows that $(b,n) = (2^8,1)$, $(2^4,2)$, $(2^2,4)$, or $(2^1,8)$. The total number of different positive integer values of b is $\boxed{4}$.

Problem 5-4



Use slopes to show that the segments marked as perpendicular are perpendicular. An altitude to the hypotenuse of the heavily outlined right triangle at the bottom of the diagram creates 2 new right triangles similar to each other and to the original triangle. Hence, the smaller shaded right triangle and the lower unshaded right triangle are similar, with ratio of similitude 2:1. If the shorter leg of the smaller of these two triangles is x, that triangle's longer leg (which is the shorter leg of the unshaded right triangle) is $2x$ and the unshaded right triangle's longer leg is $4x$. Each segment that connects a vertex to a midpoint of a non-adjoining side has a length of $x+4x = 5x$, from which the other lengths shown in the diagram are easily obtained. The area of the larger shaded triangle is $150 = (3x)(4x)/2 = 6x^2$, so the area of the smaller shaded right triangle is $(x)(2x)/2 = x^2 = \boxed{25}$.

Problem 5-5

In a regular n-gon, each exterior angle is $\frac{360}{n}$ and its adjacent interior angle is $180-\frac{360}{n}$. Similarly, in a regular m-gon, the measure of each interior angle is $180-\frac{360}{m}$. The ratio of an interior angle of M to an interior angle of N is $\frac{3}{2}$. Set $\frac{3}{2}$ equal to the complex fraction created when we form the ratio and solve. We get $360-\frac{720}{m} = 540-\frac{1080}{n}$. This simplifies to $\frac{6}{n}-\frac{4}{m} = 1$. It's clear that $n < 6$. Trying $n = 3$, 4, and 5, we get $(m,n) = \boxed{(4,3),\ (8,4),\ (20,5)}$.

Problem 5-6

If b = # of bronze medals in the box, we have that $P(\text{gold medal}) = \frac{20}{30+b}$, $P(\text{silver medal}) = \frac{10}{30+b}$, and $P(\text{bronze medal}) = \frac{b}{30+b}$. Consequently, P(2 gold medals and 2 silver medals) $= {}_4C_2 \times \left(\frac{20}{30+b}\right)^2 \times \left(\frac{10}{30+b}\right)^2$, and P(1 gold, 1 silver, 2 bronze medals) $= {}_2C_1 \times \left(\frac{20}{30+b}\right) \times \left(\frac{10}{30+b}\right) \times {}_4C_2 \times \left(\frac{b}{30+b}\right)^2$. If we equate these last two probabilities and solve, we get $200 = 2b^2$, so $b = \boxed{10}$.

