GEORGIA MATHEMATICS LEAGUE

P.O. Box 12014, Columbus, Georgia 31917-2014

All official participants must take this contest at the same time.

Contest Number 6 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **March 15, 2016**

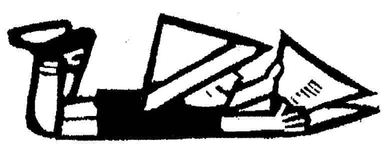
Name ______________ **Teacher** __________ **Grade Level** _____ **Score** ____

Time Limit: 30 minutes FINAL CONTEST OF THE YEAR *Answer Column*

6-1. What is the least possible area of each of two rectangles which have equal areas and integral side-lengths, but are not congruent?

6-1.

6-2. What is the length of the hypotenuse of a right triangle, whose legs have lengths $\frac{\pi}{3}$ and $\frac{\pi}{4}$?



6-2.

6-3. For what value of k do $x^3+kx^2-3x+4 = 0$ and $x^3+kx^2-5x+8 = 0$ have a common solution?

6-3.

6-4. In polygon P at the right, every angle is 45°, 90°, or 135°. If the length of every segment is 1, what is the area of P?

6-4.

6-5. If $0° < x \le 2016°$, how many angles x satisfy

$$\sin^2 2016° + \sin^2 x = 1?$$

6-5.

6-6. Pat and Lee alternately toss two fair dice, Pat tossing first. The first to roll a 9 wins the money paid by both players. If Lee pays $400 to play, how many dollars should Pat pay to make this game fair? [In a *fair game*, each player's expected value is 0 (no net gain or loss).]

6-6.

Problem 6-1

The rectangles are not congruent, so the answer is the least positive integer that can be factored in more than one way. Since $4 = 1\times 4 = 2\times 2$, the area of each rectangle is $\boxed{4}$.

Problem 6-2

By the Pythagorean Theorem, the square of the hypotenuse equals $\left(\frac{\pi}{3}\right)^2 + \left(\frac{\pi}{4}\right)^2 = \frac{25\pi^2}{144}$. To get the length of the hypotenuse, take the square root of the result. That equals $\boxed{\frac{5\pi}{12}}$.

Problem 6-3

If we call the common solution x_1, then it follows that $x_1^3+kx_1^2-3x_1+4 = x_1^3+kx_1^2-5x_1+8$, from which we subtract to see that $x_1 = 2$. Substitute back into the first of the original equations to get $2^3+k(2^2)-3(2)+4 = 0$, from which $k = \boxed{\frac{-3}{2}}$.

Problem 6-4

Method I: Draw vertical line segments connecting each vertex of the polygon to the vertex directly above or below it, thereby creating 10 rhombuses, each with sides of length 1. Dropping an altitude of one of the rhombuses creates an isosceles right triangle whose legs each have length $\frac{\sqrt{2}}{2}$. Finally, the polygon's area is equal to $10bh = 10(1)\left(\frac{\sqrt{2}}{2}\right) = \boxed{5\sqrt{2}}$.

Method II: The total area of all ten rhombuses is equal to $10\times(1)(1)(\sin 45°) = 5\sqrt{2}$.

Problem 6-5

Since $\sin 2016° = -\sin 36° = -\cos 54°$, it follows that $\sin^2 2016° = \cos^2 54°$. Rewrite the original equation as $\cos^2 54° + \sin^2 x = 1$, which has 2 sets of solutions. In one set, with n an integer, we have $x = 54+180n$, $0 \le n \le 10$. There are 11 solutions in this set. In the other set, also with n an integer, we have $x = 126+180n$, $0 \le n \le 10$. There are also 11 solutions in this set. Altogether, the total number of solutions is $11 + 11 = \boxed{22}$.

Problem 6-6

Let p = the probability that the first person wins. The probability that the first person wins by rolling a 9 on the first roll is $\frac{4}{36} = \frac{1}{9}$. The probability that he does not roll a 9 on his first turn is $\frac{8}{9}$. If the first person does not roll a 9 on his first turn, then the second person, at his first turn, will have probability p of winning, since he will then have the same chance of winning as the first person had on his first turn. The probability that the second person wins is therefore the product of the probability that the first person does not roll a 9 times the probability that the second person wins when the first person does not roll a 9 on his first roll $= \frac{8}{9}p$. Solving $p + \frac{8}{9}p = 1$, we get $p = \frac{9}{17}$. The ratio of the probability that the first person wins to the probability that the second wins is 9:8 = 450:400, so the number of dollars that the first person should pay is $\boxed{450 \text{ or } \$450}$.