# AP Review Topic 4: Curve Sketching, Derivative Graphs, & Particle Motion

**I. Curve Sketching** Concepts to Review: **1)** First Derivative Test **2)** "Concavity" Test **3)** 2nd Derivative Test

**First Derivative Test:** Use derivative function $f'(x)$ and the derivative sign line to determine location(s) of relative maximum(s) and/or minimum(s) of the graph as well as intervals of increase and intervals of decrease for $f(x)$. **Steps: 1)** Find $f'(x)$ **2)** Find critical points by setting numerator and denominator of $f'(x) = 0$. **3)** Place critical points on $f'(x)$ sign line, and test points on each subinterval with $f'(x)$ to determine interval increase/decrease.

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**1)** At what values of $x$ does $f(x) = x - 2x^{2/3}$ have a relative minimum?

(A) $\frac{64}{27}$ (B) $\frac{16}{9}$ (C) $\frac{4}{3}$ (D) 2

**"Concavity" Test**: Use the 2nd derivative function $f''(x)$and the 2nd derivative sign line to determine location of Points of Inflection(s) (POI's) as well as intervals of concave up and/or concave down for the graph of $f(x)$.
**Steps: 1)** Find $f''(x)$ **2)** Find critical points by setting numerator and denominator of $f''(x) = 0$. **3)** Place critical points on $f''(x)$ sign line, and test points on each subinterval with $f''(x)$ to determine interval(s) concave up/down **4)** Point of Inflection (POI) exists if and only if $f''(x)$ changes signs on either side of critical point.

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**2)** The graph of $y = x^4 - 2x^3$ has a point of inflection at

(A) $(0,0)$ only

(B) $(0,0)$ and $(1,-1)$

(C) $(1,-1)$ only

(D) $(0,0)$ and $(\frac{3}{2}, -\frac{27}{16})$

**2nd Derivative Test**: Use the 2nd derivative function $f''(x)$ to determine relative maximum(s) and/or minimum(s) of f(x). **Steps: 1)** Find $f'(x)$, then set $f'(x) = 0$ to find critical point(s). **2)** Find $f''(x)$, and then use critical points found from $f'(x)$ to plug INTO $f''(x)$. **3)** For critical point $x = c$, if $f''(c) < 0$, then graph is **concave Down,** therefore **relative Maximum** exists at x = c. **4)** If $f''(c) > 0$, then graph is concave Up, therefore relative Minimum exists at x = c.

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**3)** If $f'(5) = 0$, and $f''(5) = -3$, then decide if there is a relative min., relative max. or neither at $x = 5$ ? Justify your reasoning.

**II. <u>Derivative Graphs</u>:** <u>Concepts & Skills to Practice</u> **1)** Given Derivative Graph $f'(x)$: find relative max, relative min, intervals increase, intervals decrease, Points of Inflection(s), and intervals of concave up/down of $f(x)$.
**2)** Given Derivative Graph $f''(x)$: Find Points of Inflection (x-intercepts) and intervals of concave up/down of $f(x)$.

**<u>Derivative Graph $f'(x)$</u>** Steps: **1)** Create $f'(x)$ slope sign line. **2)** Label region above x-axis as positive slope (up arrow) and region below x-axis as negative slope (down arrow). **3)** The x-intercepts on the $f'(x)$ graph are potential relative max/mins of the $f'(x)$ *graph*, based on your sign line. 4) Peaks/Valleys of your $f'(x)$ graph are the Points of Inflection (POI's) of your $f(x)$ *graph.* **5)** Intervals of Increase/Decrease of your $f'(x)$ graph are intervals of concave up/concave down.

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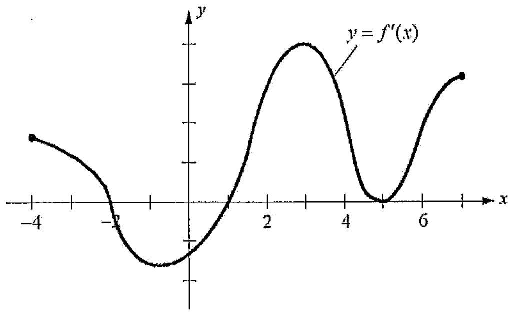
4) Given the $f'(x)$ graph below. Identify the following characteristics of the $f(x)$ graph.



***$f'(x)$ slope sign line***:

***$f''(x)$ concavity sign line***:

a) Interval(s) $f(x)$ is increasing: ______________________

b) Interval(s) $f(x)$ is decreasing: ______________________

c) Relative max of $f(x)$: x-value(s) ______________________

d) Relative min of $f(x)$: x-value(s) ______________________

e) Points of Inflection(s) of $f(x)$: ______________________

f) Interval(s) $f(x)$ is concave up: ______________________

g) Interval(s) $f(x)$ is concave down: ______________________

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**III. <u>Particle Motion</u>:** Motion along the x-axis involving position function $x(t)$, velocity $v(t)$, & acceleration $a(t)$
**<u>Steps:</u> 1)** Set v(t) = 0 to find critical point values of time ***t*** where the object is at rest(motionless) **2)** Create $v(t)$ sign line and place critical points on sign line. **3)** Test sub-intervals on sign line and indicate " + " for moving right or " - " for moving left. **4)** Acceleration $a(t)$ is the derivative of $v(t)$ **5)** If $a(t) > 0$, then velocity is increasing. If $a(t) < 0$, then velocity is decreasing. **6)** If $v(t)$ *and* $a(t)$ have the same signs, the particle's speed is increasing. If $v(t)$ *and* $a(t)$ have the opposite signs, the particle's speed is decreasing.

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5) The position of a particle moving along a line is given by $s(t) = t^3 - 12t^2 + 21t + 10$ for $t \geq 0$. For what value of $t$ is the speed of the particle increasing?

(A) $1 < t < 7$ only (B) $4 < t < 7$ only (C) $0 < t < 1$ and $4 < t < 7$ (D) $1 < t < 4$ and $t > 7$

Key

# AP Review Topic 4: Curve Sketching, Derivative Graphs, & Particle Motion

I. **Curve Sketching** Concepts to Review: **1)** First Derivative Test **2)** "Concavity" Test **3)** 2nd Derivative Test

**First Derivative Test:** Use derivative function $f'(x)$ and the derivative sign line to determine location(s) of relative maximum(s) and/or minimum(s) of the graph as well as intervals of increase and intervals of decrease for $f(x)$. **Steps: 1)** Find $f'(x)$ **2)** Find critical points by setting numerator and denominator of $f'(x) = 0$. **3)** Place critical points on $f'(x)$ sign line, and test points on each subinterval with $f'(x)$ to determine interval increase/decrease.

**1)** At what values of $x$ does $f(x) = x - 2x^{2/3}$ have a relative minimum?

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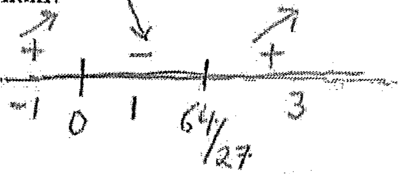


*set numerator and denominator = 0.

$f'(x) = 1 - 2\cdot\frac{2}{3}x^{-1/3}$

$0 = 1 - \frac{4}{3x^{1/3}} = \frac{3x^{1/3}-4}{3x^{1/3}}$

$3x^{1/3} - 4 = 0$ | $3x^{1/3} = 0$

$x^{1/3} = 4/3$ | $x = 0$

$x = \frac{64}{27}$

Rel. min at $x = \frac{64}{27}$ b/c $f'(x)$ changes from − to +

**"Concavity" Test**: Use the 2nd derivative function $f''(x)$and the 2nd derivative sign line to determine location of Points of Inflection(s) (POI's) as well as intervals of concave up and/or concave down for the graph of $f(x)$. **Steps: 1)** Find $f''(x)$ **2)** Find critical points by setting numerator and denominator of $f''(x) = 0$. **3)** Place critical points on $f''(x)$ sign line, and test points on each subinterval with $f''(x)$ to determine interval(s) concave up/down

**2)** The graph of $y = x^4 - 2x^3$ has a point of inflection at

(A) (0,0) only

[✓ (B)] (0,0) and (1,−1)

(C) (1,−1) only

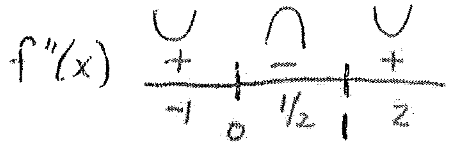
(D) (0,0) and $(\frac{3}{2}, -\frac{27}{16})$

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x$

$0 = 12x(x-1)$

$x = 0,\ x = 1$



$f(0) = 0$

$f(1) = -1$

POI at (0,0) and (1,−1) b/c $f''(x)$ change signs.

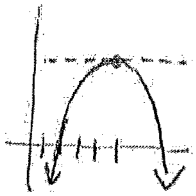
*Careful: Just because critical points at $x=0$ and $x=1$ doesn't guarantee these are POI. Be sure to confirm using sign line.

**2nd Derivative Test**: Use the 2nd derivative function $f''(x)$ to determine relative maximum(s) and/or minimum(s) of f(x). **Steps: 1)** Find $f'(x)$, then set $f'(x) = 0$ to find critical point(s). **2)** Find $f''(x)$, and then use critical points found from $f'(x)$ to plug INTO $f''(x)$. **3)** For critical point $x = c$, if $f''(c) < 0$, then graph is **concave Down**, therefore **relative Maximum** exists at x = c. **4)** If $f''(c) > 0$, then graph is concave Up, therefore relative Minimum exists at x = c.

**3)** If $f'(5) = 0$, and $f''(5) = -3$, then decide if there is a relative min., relative max. or neither at $x = 5$ ? Justify your reasoning.

$f'(5) = 0$ means the slope of graph at $x=5$ is zero (horizontal)

$f''(5) = -3$ means the graph is concave down because $f''(5) < 0$



Rel. max at $x = 5$ b/c $f'(5) = 0$ and $f''(5) < 0$ (2nd Derivative Test)

**II. Derivative Graphs:** Concepts & Skills to Practice **1)** Given Derivative Graph $f'(x)$: find relative max, relative min, intervals increase, intervals decrease, Points of Inflection(s), and intervals of concave up/down of $f(x)$.
**2)** Given Derivative Graph $f''(x)$: Find Points of Inflection (x-intercepts) and intervals of concave up/down of $f(x)$.

**Derivative Graph $f'(x)$** Steps: **1)** Create $f'(x)$ slope sign line. **2)** Label region above x-axis as positive slope (up arrow) and region below x-axis as negative slope (down arrow). **3)** The x-intercepts on the $f'(x)$ graph are potential relative max/mins of the $f'(x)$ *graph*, based on your sign line. 4) Peaks/Valleys of your $f'(x)$ graph are the Points of Inflection (POI's) of your $f(x)$ *graph*. **5)** Intervals of Increase/Decrease of your $f'(x)$ graph are intervals of concave up/concave down.

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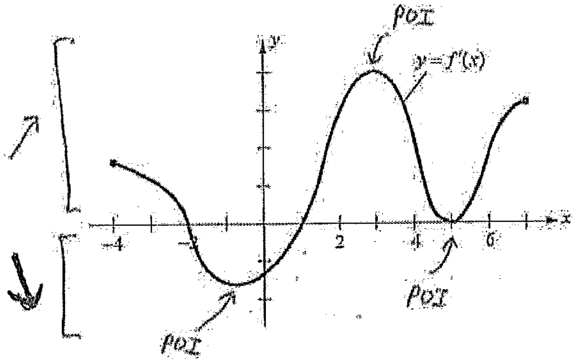
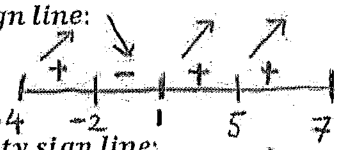
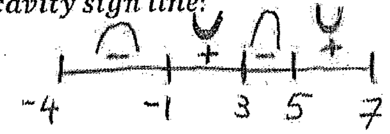
4) Given the $f'(x)$ graph below. Identify the following characteristics of the $f(x)$ graph.



$f'(x)$ ***slope sign line:***



$f''(x)$ ***concavity sign line:***



a) Interval(s) $f(x)$ is increasing: $(-4,-2) \cup (1,5) \cup (5,7)$ b/c $f'(x) > 0$

b) Interval(s) $f(x)$ is decreasing: $(-2,1)$ b/c $f'(x) < 0$

c) Relative max of $f(x)$: x-value(s) $x = -2$ b/c $f'(x)$ changes from + to −

d) Relative min of $f(x)$: x-value(s) $x = 1$ b/c $f'(x)$ changes from − to +

e) Points of Inflection(s) of $f(x)$: POI at $x = -1, 3, 5$ b/c $f''(x)$ change signs

f) Interval(s) $f(x)$ is concave up: $(-1,3) \cup (5,7)$ b/c $f''(x) > 0$

g) Interval(s) $f(x)$ is concave down: $(-4,-1) \cup (3,5)$ b/c $f''(x) < 0$

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**III. Particle Motion:** Motion along the x-axis involving position function $x(t)$, velocity $v(t)$, & acceleration $a(t)$
**Steps:** **1)** Set v(t) = 0 to find critical point values of time $t$ where the object is at rest(motionless) **2)** Create $v(t)$ sign line and place critical points on sign line. **3)** Test sub-intervals on sign line and indicate " + " for moving right or " - " for moving left. **4)** Acceleration $a(t)$ is the derivative of $v(t)$ **5)** If $a(t) > 0$, then velocity is increasing. If $a(t) < 0$, then velocity is decreasing. **6)** If $v(t)$ *and* $a(t)$ have the same signs, the particle's speed is increasing. If $v(t)$ *and* $a(t)$ have the opposite signs, the particle's speed is decreasing.

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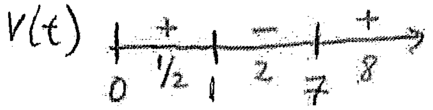
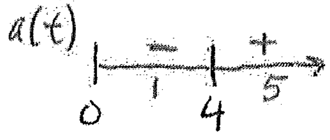
5) The position of a particle moving along a line is given by $s(t) = t^3 - 12t^2 + 21t + 10$ for $t \geq 0$. For what value of $t$ is the speed of the particle increasing? *compare signs b/t v(t) and a(t).

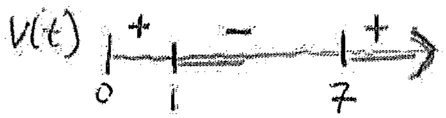
(A) $1 < t < 7$ only (B) $4 < t < 7$ only (C) $0 < t < 1$ and $4 < t < 7$ $\boxed{\text{(D) } 1 < t < 4 \text{ and } t > 7}$

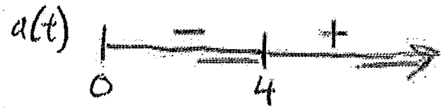
$v(t) = 3t^2 - 24t + 21$
$0 = 3(t^2 - 8t + 7)$
$0 = 3(t-7)(t-1)$
$t = 1,\ t = 7$



$a(t) = 6t - 24$
$0 = 6(t-4)$
$t = 4$



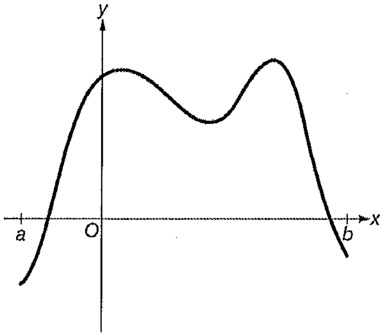




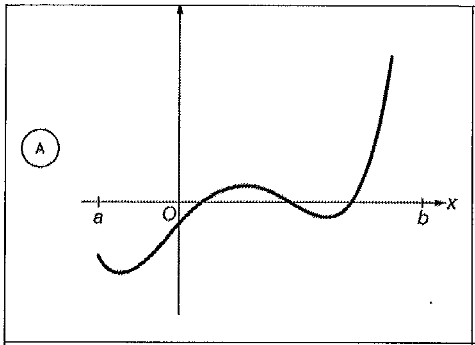
v(t) and a(t) have same signs in intervals: $(1,4) \cup (7,\infty)$

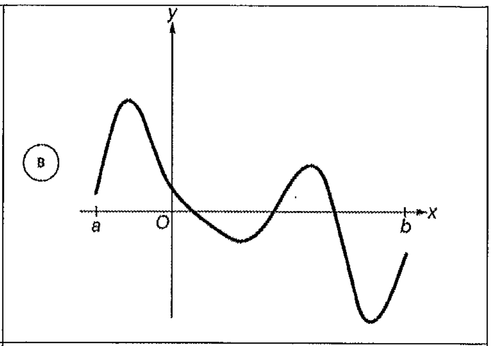
# AP Calculus AB Review Topic 4: Curve Sketching, Derivative Graphs, & Particle Motion Worksheet

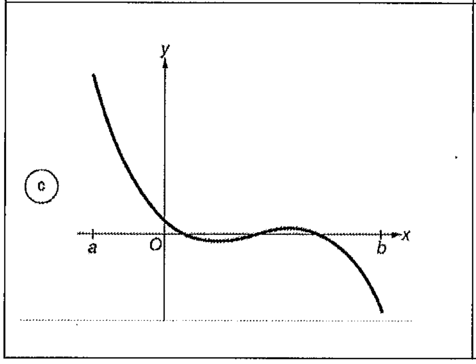
1)

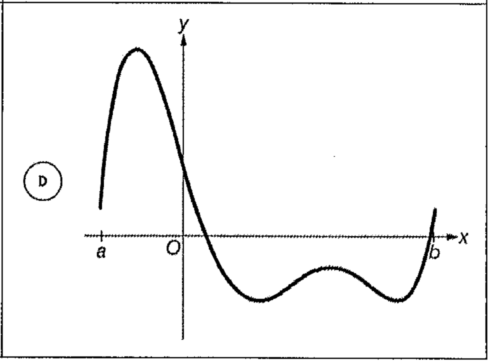


Graph of $f$

A


B


C


D


The figure above shows the graph of $f$ on the interval $[a, b]$. Which of the following could be the graph of $f'$, the derivative of $f$, on the interval $[a, b]$ ?

**2)**

In the $xy$-plane, the point $(0, -2)$ is on the curve $C$. If $\frac{dy}{dx} = \frac{4x}{9y}$ for the curve, which of the following statements is true?

- A At the point $(0, -2)$, the curve $C$ has a relative minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.
- B At the point $(0, -2)$, the curve $C$ has a relative minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.
- C At the point $(0, -2)$, the curve $C$ has a relative maximum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.
- D At the point $(0, -2)$, the curve $C$ has a relative maximum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

3) The first derivative of the function $f$ is given by $f'(x) = (x^3 + 2)\ e^x$. What is the $x$-coordinate of the inflection point of the graph of $f$ ?

(A) $-3.196$　　(B) $-1.260$　　(C) $-1$　　(D) $0$

4) Let $f$ be a twice differentiable function such that $f(1) = 7$ and $f(3) = 12$. If $f'(x) > 0$ and $f''(x) < 0$ for all real numbers $x$, which of the following is a possible value for $f(5)$ ?

(A) 16　　(B) 17　　(C) 18　　(D) 19

5) If the graph of $y = ax^3 - 6x^2 + bx - 4$ has a point of inflection at $(2, -2)$, what is the value of $a + b$ ?

(A) $-2$　　(B) 3　　(C) 6　　(D) 10

6) A particle moves along the $x$-axis so that at any time $t$, its position is given by $x(t) = 3\sin t + t^2 + 7$. What is velocity of the particle when its acceleration is zero?

(A) 1.504　　(B) 1.847　　(C) 2.965　　(D) 3.696

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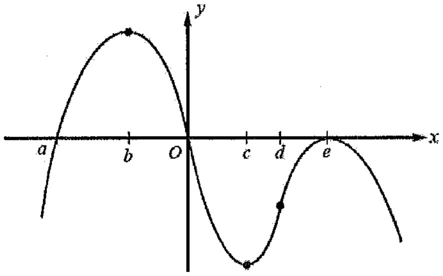
7)



Graph of $f''$

The second derivative of the function $f$ is given by $f''(x) = x(x + a)(x - e)^2$ and the graph of $f''$ is shown above. For what values of $x$ does the graph of $f$ have a point of inflection?

(A) $b$ and $c$　　(B) $b$, $c$ and $e$　　(C) $b$, $c$ and $d$　　(D) $a$ and 0

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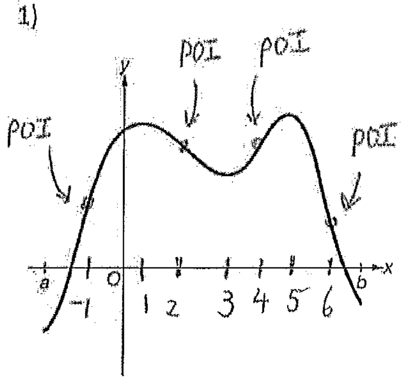
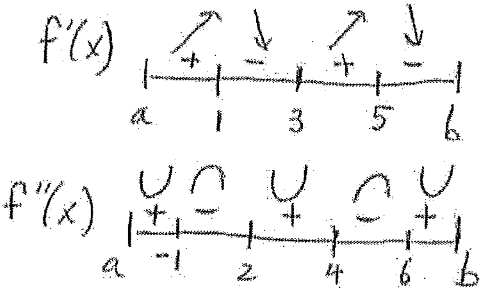
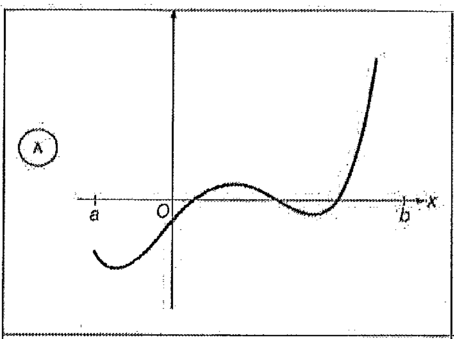
Key

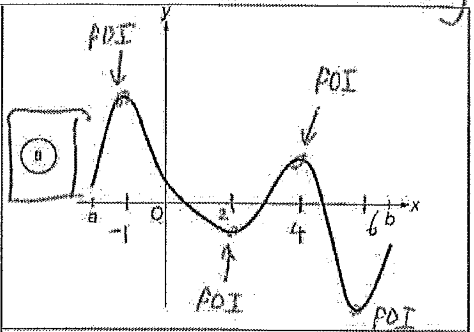
1)

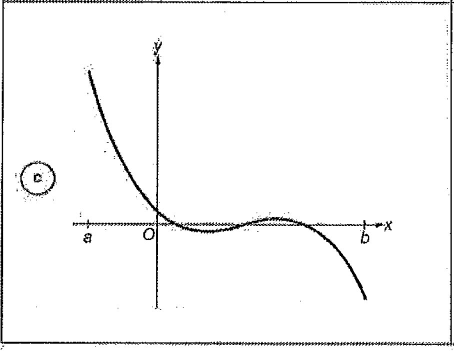


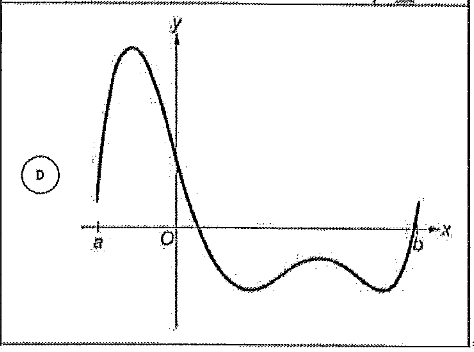
Graph of $f$











The figure above shows the graph of $f$ on the interval $[a, b]$. Which of the following could be the graph of $f'$, the derivative of $f$, on the interval $[a, b]$?

$\boxed{B}$ * There are 4 POI's, so there will be 4 peaks/valleys on the f'(x) graph

* There are 3 relative extrema, so there will be 3 x-intercepts

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2) $\frac{dy}{dx} = \frac{\overbrace{4x}^{f}}{\underbrace{9y}_{g}}$ $\quad \frac{d^2y}{dx^2} = \frac{\overbrace{(4)}^{f'}\overbrace{(9y)}^{g} - \overbrace{(4x)}^{f}\overbrace{(9\frac{dy}{dx})}^{g'}}{(9y)^2}$ $\quad \left.\frac{d^2y}{dx^2}\right|_{(0,-2)} = \frac{4(9)(-2) - 4(0)(9)\left(\frac{4(0)}{9(-2)}\right)}{(9(-2))^2} = \frac{-2}{9} < 0$

concave down

find 2nd derivative

2)

In the $xy$-plane, the point $(0, -2)$ is on the curve $C$. If $\frac{dy}{dx} = \frac{4x}{9y}$ for the curve, which of the following statements is true?

$\left.\frac{dy}{dx}\right|_{(0,-2)} \to \frac{4(0)}{9(-2)} = 0$

Ⓐ At the point $(0, -2)$, the curve $C$ has a relative minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

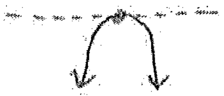
Ⓑ At the point $(0, -2)$, the curve $C$ has a relative minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Ⓒ At the point $(0, -2)$, the curve $C$ has a relative maximum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

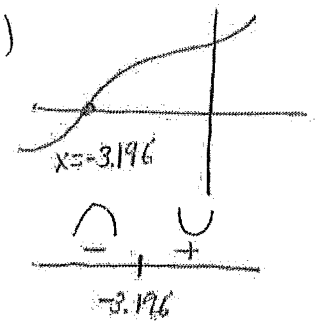
$\boxed{D}$ At the point $(0, -2)$, the curve $C$ has a relative maximum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

slope = 0
concave down
results in relative max at x = 0

* ↗ 2nd Derivative Test.



**3)** The first derivative of the function $f$ is given by $f'(x) = (\overset{f}{\overbrace{x^3+2}})\,\overset{g}{\overbrace{e^x}}$. What is the $x$-coordinate of the inflection point of the graph of $f$ ?



$\boxed{\text{(A) } -3.196}$ (B) $-1.260$ (C) $-1$ (D) 0

$$f''(x) = \overset{f'}{\overbrace{(3x^2)}}\overset{g}{\overbrace{(e^x)}} + \overset{f}{\overbrace{(x^3+2)}}\overset{g'}{\overbrace{(e^x)}}$$

$$0 = e^x(3x^2+x^3+2) \rightarrow \boxed{x \simeq -3.196}$$

**4)** Let $f$ be a twice differentiable function such that $f(1) = 7$ and $f(3) = 12$ . If $f'(x) > 0$ and $f''(x) < 0$ for all real numbers $x$ , which of the following is a possible value for $f(5)$ ?

$\boxed{\text{(A) } 16}$ (B) 17 (C) 18 (D) 19

i) If $f'(x) > 0$, then slope between the points should be positive

ii) If $f''(x) < 0$, then slope should decreasing (slope decreases from 2.5 to 2 when $x = 16$)

| X | y |
|---|---|
| 1 | 7 |
| 3 | 12 |
| 5 | 16 |

$> \text{slope} = \frac{12-7}{3-1} = \frac{5}{2} = 2.5$

$> \text{slope} = \frac{16-12}{5-3} = \frac{4}{2} = 2$

**5)** If the graph of $y = ax^3 - 6x^2 + bx - 4$ has a point of inflection at $(2, -2)$ , what is the value of $a+b$ ?

find $f''(x)$, set $f''(x) = 0$

(A) $-2$ (B) 3 (C) 6 $\boxed{\text{(D) } 10}$

$y' = 3ax^2 - 12x + b$

$y'' = 6ax - 12$

$x=2$

$0 = 6ax - 12$

$0 = 6a(2) - 12$

$12 = 12a$ $\boxed{a=1}$

$y = ax^3 - 6x^2 + bx - 4$

$-2 = 1(2)^3 - 6(2)^2 + b(2) - 4$

$-2 = 8 - 24 + 2b - 4$

$18 = 2b$ $\boxed{b=9}$

$a+b = 9+1 = \boxed{10}$

**6)** A particle moves along the $x$-axis so that at any time $t$ , its position is given by $x(t) = 3\sin t + t^2 + 7$ . What is velocity of the particle when its acceleration is zero? set $a(t) = 0$

(A) 1.504 (B) 1.847 (C) 2.965 $\boxed{\text{(D) } 3.696}$

$v(t) = 3\cos(t) + 2t$

$a(t) = -3\sin(t) + 2$

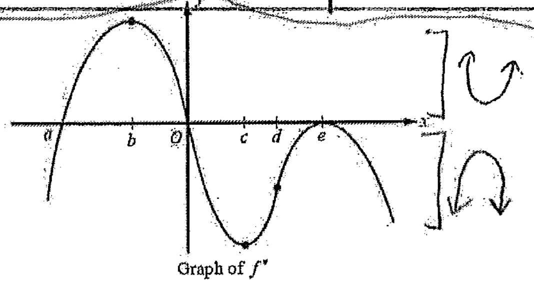
$0 = -3\sin(t) + 2$

$t \simeq 0.729$

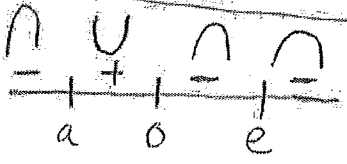
$v(0.729) = 3\cos(0.729) + 2(0.729)$

$\boxed{V(0.729) = 3.696}$

**7)**



Graph of $f''$



The second derivative of the function $f$ is given by $f''(x) = x(x+a)(x-e)^2$ and the graph of $f''$ is shown above. For what values of $x$ does the graph of $f$ have a point of inflection?

(A) $b$ and $c$ (B) $b$ , $c$ and $e$ (C) $b$ , $c$ and $d$ $\boxed{\text{(D) } a \text{ and } 0}$

POI at $x=a$ and $x=0$ since $f''(x)$ change signs.

## Optimization Review Problem (Involving Cost)

1)

A rectangular storage container with an open top is to have a Volume of 10 $m^3$. The length of its base is twice its length. Material for the base costs \$10/ $m^2$. Material for the sides cost \$6/$m^2$. Find the cost of material for the cheapest container. (Hint: Minimize surface area)

2)

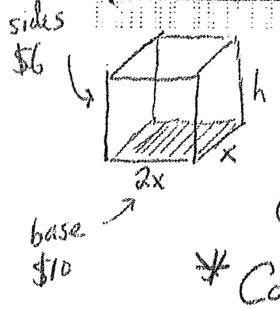
The manager of a department store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$14 per running foot. The forth side will be built of cement blocks, at a cost of \$28 per running foot. What dimensions will minimize the total cost of the building materials? What will this minimum cost be?

Key

## Optimization Review Problem (Involving Cost)

**1)**

A rectangular storage container with an open top is to have a Volume of 10 m$^3$. The length of its base is twice its length. Material for the base costs \$10/ m$^2$. Material for the sides cost \$6/m$^2$. Find the cost of material for the cheapest container. (Hint: Minimize surface area)



$S = 2x^2 + xh + xh + 2xh + 2xh$

$S = 2x^2 + 6xh$

$Cost = 10(2x^2) + 6(6xh)$

$* \; Cost = 20x^2 + 36xh \leftarrow$

$Volume = (2x)(x)(h)$

$10 = 2x^2 h$

$\frac{10}{2x^2} = h$

$\boxed{\frac{5}{x^2} = h}$

$Cost = 20x^2 + 36x\left(\frac{5}{x^2}\right)$

$C(x) = 20x^2 + \frac{180}{x}$

$C(x) = 20x^2 + 180x^{-1}$

$C'(x) = 40x - 180x^{-2}$

$0 = 40x - \frac{180}{x^2}$

$\frac{180}{x^2} = \frac{40x}{1}$

$40x^3 = 180$

$x^3 = \frac{180}{40} = \frac{9}{2}$

$x^3 = 9/2$

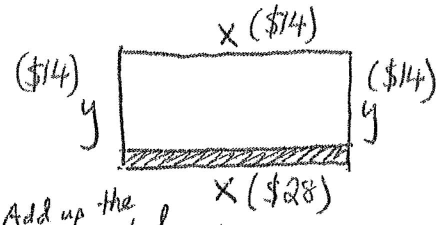
$x = \sqrt[3]{9/2} \approx 1.651$ meters

$C(1.651) = 20(1.651)^2 + \frac{180}{1.651}$

$= \boxed{\$163.54}$

**2)**

The manager of a department store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$14 per running foot. The forth side will be built of cement blocks, at a cost of \$28 per running foot. What dimensions will minimize the total cost of the building materials? What will this minimum cost be? (perimeter)



Add up the costs associated with each side

$* \; Cost = 28x + 14x + 14y + 14y$

$C = 42x + 28y \leftarrow$

$Area = xy$

$600 = xy$

$\boxed{\frac{600}{x} = y}$

$C(x) = 42x + 28\left(\frac{600}{x}\right)$

$C(x) = 42x + 16800x^{-1}$

$C'(x) = 42 - 16800x^{-2}$

$0 = 42 - \frac{16800}{x^2}$

$\frac{16800}{x^2} = \frac{42}{1}$

$42x^2 = 16800$

$x^2 = \frac{16800}{42}$

$x^2 = 400$

$x = 20$ ft

$y = \frac{600}{x} \rightarrow \frac{600}{20} = 30$ft

$Cost = 42(20) + 28(30)$

$\boxed{Cost = \$1680}$