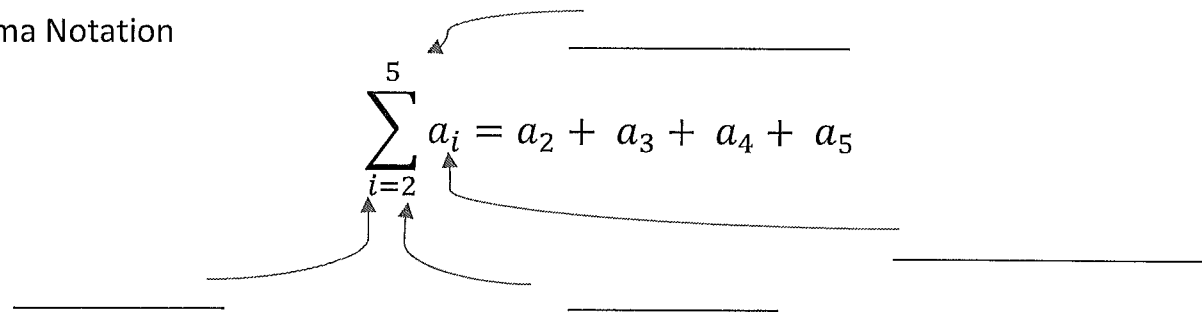


**Ch. 4.2a Notes**

**I. Sigma Notation**



Ex.1  $\sum_{i=2}^4 i^2 + 1 =$

**II. Summation Formulas:**

1)  $\sum_{i=1}^n 1 = n$

2)  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

3)  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

4)  $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

5)  $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$

**Example 2**

$$\sum_{i=1}^8 (3i^2 + 2) =$$

**Example 3**

$$\sum_{i=1}^{10} (i + 2)^2 =$$

**Example 4**

$$\sum_{k=1}^n \frac{1}{n} (k^2 - 1) =$$

2

### III. Limits as $n$ approaches infinity

\*Think back about finding horizontal asymptotes

**Example 5:** If  $S(n) = \frac{1}{n^2} \left[ \frac{n(n+1)}{2} \right]$ , then find  $\lim_{n \rightarrow \infty} S(n)$

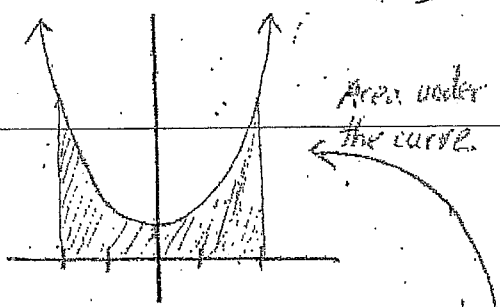
**Example 6:** Find  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{2i}{n} \right) \left( \frac{2}{n} \right)$

**Example 7:** Find  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left( 1 + \frac{2i}{n} \right)^2 \left( \frac{2}{n} \right)$

# 4.26 - Riemann Sums

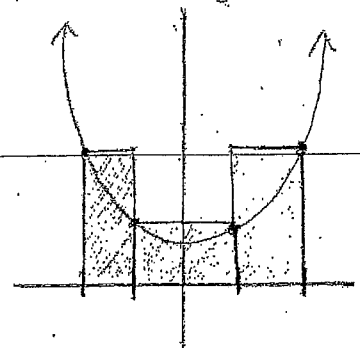
Riemann Sums - Using rectangles to estimate area of region.  
(Area under a curve)

Consider the function  
 $f(x) = x^2 + 1$   $[-2, 2]$



Suppose we want to estimate the area of the shaded region using a given number of rectangles.

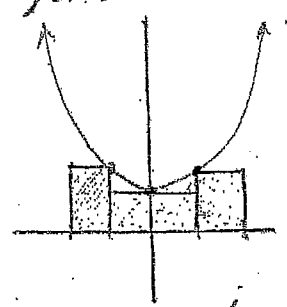
1) Upper rectangles or Circumscribed rectangles



\* Using these rectangles will provide an overestimation of the area under the curve.

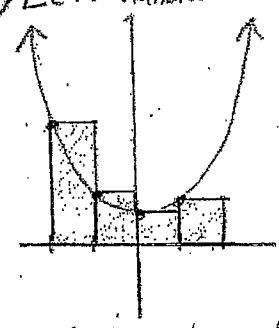
\* Notice that one corner of each rectangle is on the graph. This ensures that the height of the rectangle is the same as the value of the function at the point where they connect.

2) Lower or Inscribed rectangles



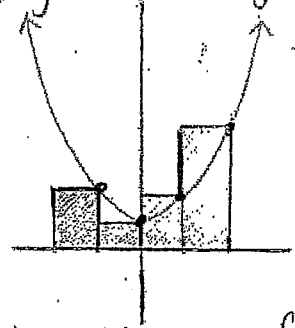
These rectangles provide an underestimation of the area under the curve

3) Left-handed Rectangles



The left corner of each rectangle attaches to the graph.

4) Right-handed Rectangles

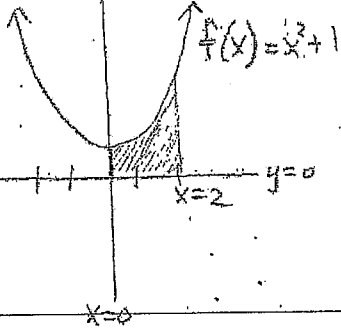


The right corner of each rectangle attaches to the graph.

4.2b (continued)

Ex. 1

Use 4 rectangles to estimate upper and lower sums for the area bounded by  $x=0$ ,  $y=0$ ,  $x=2$ , and  $y=x^2+1$



Step 1: Determine width of each rectangle.

$$\text{width} = \frac{b-a}{n}$$

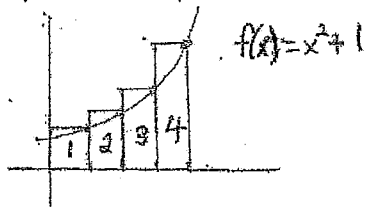
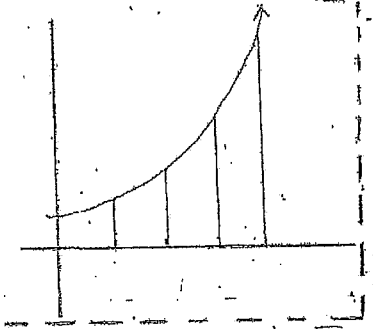
$a$  = left endpoint

$b$  = right endpoint

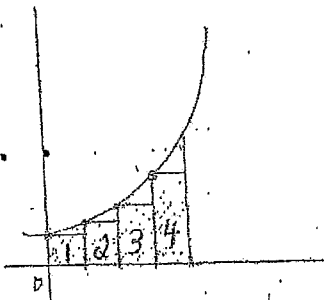
$n$  = number of rectangles

Step 2: Draw the graph. Section off each interval.

Step 3: Find sum of areas of appropriate rectangles.



b) Find lower sum

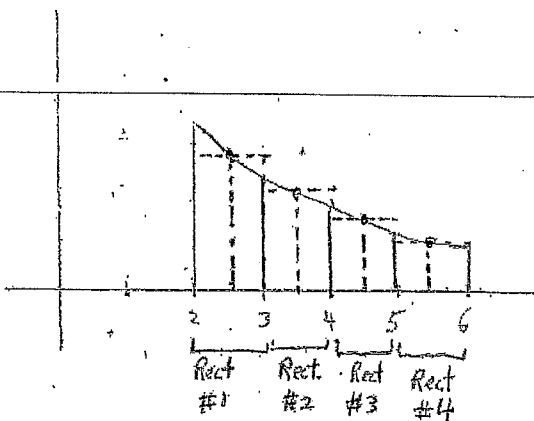


4.2b (continued)

Midpoint Rule: Similar to upper/lower sum but use the midpoint of each rectangle to calculate rectangle's height.

**Ex. 2** Estimate area under curve  $f(x) = \frac{8}{x^2+1}$  from  $[2, 6]$   
Use midpoint rule with 4 subintervals.

$$\text{width} = \frac{b-a}{n} = \frac{6-2}{4} = \frac{4}{4} = 1$$



• Why would midpoint sum be a better approximation of area than upper or lower sum?

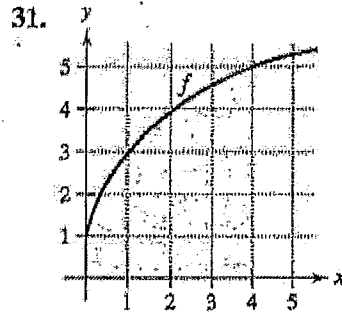
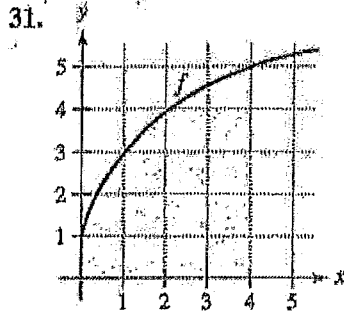
This is because each rectangle has portions above and below the graph.

\* Note: Midpoint sum is not the average between upper and lower sum!

6

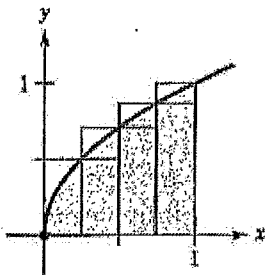
Ch. 4.2 Homework Problems

Using Upper and Lower Sums In Exercises 31 and 32, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

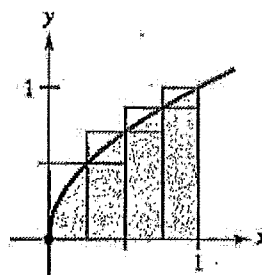


Finding Upper and Lower Sums for a Region In Exercises 33–36, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

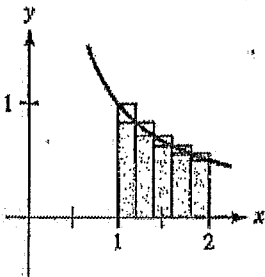
33.  $y = \sqrt{x}$



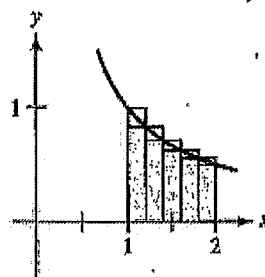
33.  $y = \sqrt{x}$



35.  $y = \frac{1}{x}$



35.  $y = \frac{1}{x}$



8

Ch. 4.2 Continued

Approximating the Area of a Plane Region In Exercises 25–30, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the  $x$ -axis over the given interval.

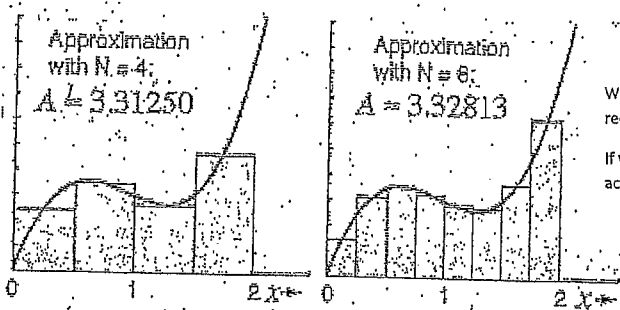
25.  $f(x) = 2x + 5$ ,  $[0, 2]$ , 4 rectangles

27.  $g(x) = 2x^2 - x - 1$ ,  $[2, 5]$ , 6 rectangles

29.  $f(x) = \cos x$ ,  $\left[0, \frac{\pi}{2}\right]$ , 4 rectangles



# 4.2c Finding Exact Area using limits



We can continually improve the Area Approximation under the curve by increasing the number of rectangles: above (n = 4) and n = 8, ... n = 16...

If we let n go out to infinity, (using limits) we will have something better than an approximation, we will achieve the actual area under the curve.

$$\begin{aligned}
 \text{Exact Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \left( \frac{b-a}{n} \right) \cdot f \left( a + \left( \frac{b-a}{n} \right) i \right) \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (\text{width}) \cdot f(\text{left endpoint} + \text{width} \cdot i)
 \end{aligned}$$

Memorize → "width + left plus width times i"

**Ex. 1** Find exact area between  $f(x) = 4 - x^2$  and x-axis from  $[-2, 2]$

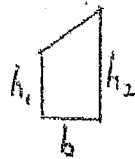
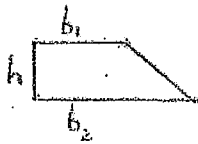
10

## 4.6 Trapezoids

\* Better approximation than inscribed, circumscribed, or midpoint rectangles.

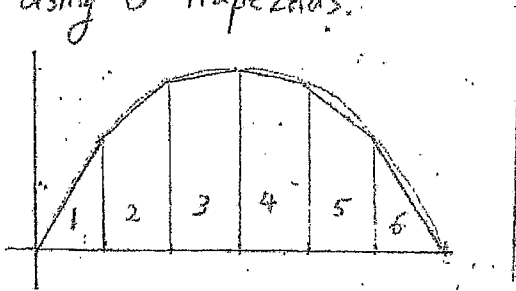
Trapezoidal Rule: Approximate Area of region using areas of trapezoids.

Review: Area of Trapezoid =  $\frac{1}{2}h(b_1+b_2)$  or  $\frac{h}{2}(b_1+b_2)$



$$\text{Area} = \frac{b}{2}(h_1+h_2) \text{ or } \frac{1}{2}w(h_1+h_2)$$

**Ex. 1** Estimate area bounded by  $f(x) = 6x - x^2$  and the x-axis using 6 trapezoids.



Limit Definition of Area Practice Problems WS

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{b-a}{n} \cdot f \left[ a + \frac{b-a}{n} i \right] \right]$$

1)  $y = 2x^2 - 3x + 2$   $[1, 3]$

2)  $y = 1 - 2x - x^2$   $[-1, 4]$

2

Use Limit Definition of Area:  $\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{b-a}{n} \cdot f \left[ a + \frac{b-a}{n} i \right] \right]$

1)  $y = 2x^2 - 3x + 2$  [1, 3]

width =  $\frac{3-1}{n} = \frac{2}{n}$

$\lim_{n \rightarrow \infty} \sum \frac{2}{n} \cdot f \left[ 1 + \frac{2}{n} i \right]$

$\lim_{n \rightarrow \infty} \sum \frac{2}{n} \left[ 2 \left( 1 + \frac{2}{n} i \right)^2 - 3 \left( 1 + \frac{2}{n} i \right) + 2 \right]$

$\frac{2}{n} \left[ 2 \left( 1 + \frac{2}{n} i \right) \left( 1 + \frac{2}{n} i \right) - 3 \left( 1 + \frac{2}{n} i \right) + 2 \right]$

$\frac{2}{n} \left[ 2 \left( 1 + \frac{2}{n} i + \frac{2}{n} i + \frac{4}{n^2} i^2 \right) - 3 \left( 1 + \frac{2}{n} i \right) + 2 \right]$

$\frac{2}{n} \left[ 2 \left( 1 + \frac{4}{n} i + \frac{4}{n^2} i^2 \right) - 3 \left( 1 + \frac{2}{n} i \right) + 2 \right]$

$\frac{2}{n} \left[ 2 + \frac{8}{n} i + \frac{8}{n^2} i^2 - 3 - \frac{6}{n} i + 2 \right]$

$\sum \frac{2}{n} \left[ \frac{8}{n^2} i^2 + \frac{2}{n} i + 1 \right]$

$\sum \frac{16}{n^3} i^2 + \frac{4}{n^2} i + \frac{2}{n}$

$\sum \frac{16}{n^3} i^2 + \sum \frac{4}{n^2} i + \sum \frac{2}{n}$

$\frac{16}{n^3} \left[ \sum i^2 \right] + \frac{4}{n^2} \left[ \sum i \right] + \frac{2}{n} \left[ \sum 1 \right]$

$\lim_{n \rightarrow \infty} \frac{16}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right] + \frac{4}{n^2} \left[ \frac{n(n+1)}{2} \right] + \frac{2}{n} [n]$

$\lim_{n \rightarrow \infty} \frac{32n^3 + \dots}{6n^3} + \frac{4n^2 + \dots}{2n^2} + \frac{2n}{n}$

$\frac{32}{6} + 2 + 2 = \frac{28}{3}$

2)  $y = 1 - 2x - x^2$  [-1, 4]

width =  $\frac{4-(-1)}{n} = \frac{5}{n}$

$\lim_{n \rightarrow \infty} \sum \frac{5}{n} \cdot f \left[ -1 + \frac{5}{n} i \right]$

$\sum \frac{5}{n} \left[ 1 - 2 \left( -1 + \frac{5}{n} i \right) - \left( -1 + \frac{5}{n} i \right)^2 \right]$

$\sum \frac{5}{n} \left[ 1 + 2 - \frac{10}{n} i - \left( -1 + \frac{5}{n} i \right) \left( -1 + \frac{5}{n} i \right) \right]$

$\sum \frac{5}{n} \left[ 3 - \frac{10}{n} i - \left( 1 - \frac{5}{n} i - \frac{5}{n} i + \frac{25}{n^2} i^2 \right) \right]$

$\sum \frac{5}{n} \left[ 3 - \frac{10}{n} i - \left( 1 - \frac{10}{n} i + \frac{25}{n^2} i^2 \right) \right]$

$\sum \frac{5}{n} \left[ 3 - \frac{10}{n} i - 1 + \frac{10}{n} i - \frac{25}{n^2} i^2 \right]$

$\sum \frac{5}{n} \left[ 2 - \frac{25}{n^2} i^2 \right]$

$\sum \frac{10}{n} - \frac{125}{n^3} i^2 = \sum \frac{10}{n} - \sum \frac{125 i^2}{n^3}$

$\frac{10}{n} \left[ \sum 1 \right] - \frac{125}{n^3} \left[ \sum i^2 \right]$

$\lim_{n \rightarrow \infty} \frac{10}{n} [n] - \frac{125}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right]$

$\lim_{n \rightarrow \infty} \frac{10n}{n} - \frac{250n^3 + \dots}{6n^3}$

$10 - \frac{250}{6} = \frac{-95}{3}$

Riemann Sum Worksheet

Name \_\_\_\_\_

For each problem sketch the graph showing the appropriate region, then approximate the area bound by the curve and the x-axis on the given interval using 6 different Riemann sums Left, Right, Upper, Lower, midpoint, Trapezoidal, using the specified number of subintervals.

- Function :  $f(x) = -(x - 3)^2 + 20$  on Interval  $[0, 5]$  using 5 subintervals

Graph:

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

- Function :  $f(x) = 2\sin x + 3$  on Interval  $[0, 2\pi]$  using 6 subintervals

Graph:

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

3. Function :  $f(x) = \sqrt[3]{2x - 1} + 5$  on Interval  $[-2, 2]$  using 4 subintervals

Graph:

4.

x	-4	-2	0	3	6	13	20
f(x)	8	12	18	4	9	31	12

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

Left Sum- 6 Subintervals	Right Sum- 6 Subintervals
Lower Sum- 6 Subintervals	Upper Sum- 6 Subintervals
Midpoint Sum- 3 Subintervals	Trapezoidal Sum- 6 Subintervals

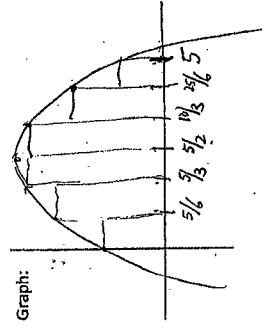
Key

Riemann Sum Worksheet

Name \_\_\_\_\_  
 For each problem sketch the graph showing the appropriate region, then approximate the area bound by the curve and the x-axis on the given interval using 6 different Riemann sums: Left, Right, Upper, Lower, midpoint, Trapezoidal, using the specified number of subintervals.

- Function:  $f(x) = -(x-3)^2 + 20$  on interval  $[0, 5]$  using 5 subintervals

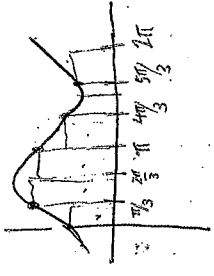
width =  $\frac{b-a}{n} = \frac{5-0}{5} = 1$



Left Sum	Right Sum
$\frac{1}{5} [f(0) + f(1) + f(2) + f(3) + f(4) + f(5)]$	$\frac{1}{5} [f(1) + f(2) + f(3) + f(4) + f(5)]$
Lower Sum	Upper Sum
$\frac{1}{5} [f(0) + f(1) + f(2) + f(3) + f(4) + f(5)]$	$\frac{1}{5} [f(1) + f(2) + f(3) + f(4) + f(5)]$
Midpoint Sum	Trapezoidal Sum
$\frac{1}{5} [f(1/2) + f(3/2) + f(5/2) + f(7/2) + f(9/2)]$	$\frac{1}{10} [h_1 + 2h_2 + 2h_3 + \dots + h_n]$ $\frac{1}{10} [f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5)]$ $+ 2f(1/2) + f(5)$

- Function:  $f(x) = 2\sin x + 3$  on interval  $[0, 2\pi]$  using 6 subintervals.  $\frac{b-a}{n} = \frac{2\pi - 0}{6} = \frac{\pi}{3}$

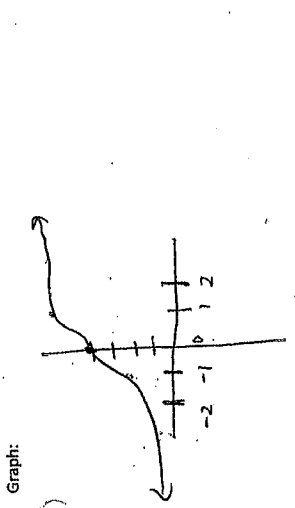
Graph:



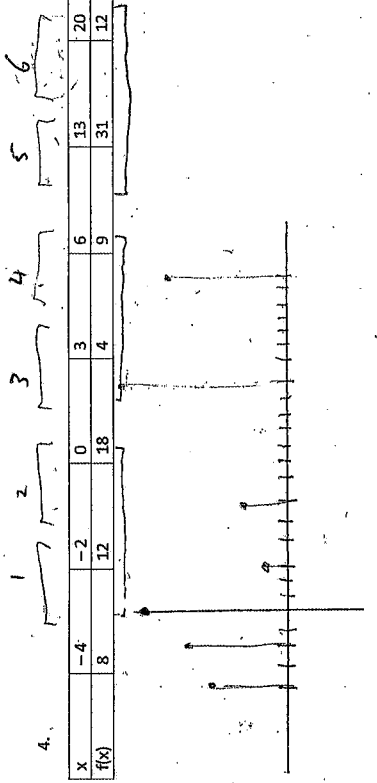
Left Sum	Right Sum
$\frac{\pi}{3} [f(0) + f(\pi/3) + f(2\pi/3) + f(\pi) + f(4\pi/3) + f(5\pi/3)]$	$\frac{\pi}{3} [f(\pi/3) + f(2\pi/3) + f(\pi) + f(4\pi/3) + f(5\pi/3)]$
Lower Sum	Upper Sum
$\frac{\pi}{3} [f(0) + f(\pi/3) + f(\pi) + f(4\pi/3) + f(5\pi/3) + f(2\pi)]$	$\frac{\pi}{3} [f(\pi/3) + f(2\pi/3) + f(\pi) + f(4\pi/3) + f(5\pi/3) + f(2\pi)]$
Midpoint Sum	Trapezoidal Sum
$\frac{\pi}{3} [f(\pi/6) + f(\pi/2) + f(5\pi/6) + f(7\pi/6) + f(3\pi/2) + f(11\pi/6)]$	$\frac{\pi}{3} [f(0) + 2f(\pi/3) + 2f(2\pi/3) + 2f(\pi) + 2f(4\pi/3) + 2f(5\pi/3) + f(2\pi)]$

6

3. Function:  $f(x) = \sqrt[3]{2x-1} + 5$  on Interval  $[-2, 2]$  using 4 subintervals  $W = \frac{b-a}{n} = \frac{2-(-2)}{4} = 1$



Left Sum $1 [f(-2) + f(-1) + f(0) + f(1)]$	Right Sum $1 [f(-1) + f(0) + f(1) + f(2)]$
Lower Sum $1 [f(-2) + f(-1) + f(0) + f(1)]$	Upper Sum $1 [f(-1) + f(0) + f(1) + f(2)]$
Midpoint Sum $1 [f(-1.5) + f(-0.5) + f(0.5) + f(1.5)]$	Trapezoidal Sum $\frac{1}{2} \cdot 1 [f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2)]$



Left Sum- 6 Subintervals $2f(-4) + 2[f(-2)] + 3f(0) + 3f(2) + 7f(4) + 2(8) + 2(12) + 3(18) + 3(4) + 7(9) + 7(31)$	Right Sum- 6 Subintervals $2f(-2) + 2f(0) + 2f(2) + 3f(4) + 3f(6) + 7f(8) + 7f(10) + 7f(12)$
Lower Sum- 6 Subintervals $2f(-4) + 2f(-2) + 3f(3) + 3f(5) + 7f(7) + 7f(9) + 7f(11)$	Upper Sum- 6 Subintervals $2f(-2) + 2f(0) + 3f(2) + 3f(4) + 3f(6) + 7f(8) + 7f(10)$
Midpoint Sum- 3 Subintervals $4f(-2) + 6f(2) + 14f(6)$ $4(12) + 6(4) + 14(31)$	Trapezoidal Sum- 6 Subintervals $\frac{2}{2} [f(-4) + f(-2)] + \frac{2}{2} [f(-2) + f(0)] + \frac{2}{2} [f(0) + f(2)] + \frac{2}{2} [f(2) + f(4)] + \frac{2}{2} [f(4) + f(6)] + \frac{2}{2} [f(6) + f(8)] + \frac{2}{2} [f(8) + f(10)] + \frac{2}{2} [f(10) + f(12)]$



Non-AP Calculus 4-2, 4-6 Riemann Sums WS: Using Tables of Values

1) Selected values of a function,  $f$ , are given in the table below:

$x$	0	5	8	9	12	18	20
$f(x)$	4	2	3	7	3	6	10

a) Give the middle approximation with 3 subintervals for  $f$  on the interval  $[0, 20]$

b) Use right-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[0, 20]$

c) Use left-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[0, 9]$

d) Use trapezoids to approximate the area with 2 subintervals for  $f$  on the interval  $[0, 20]$

2) Selected values of a function,  $f$ , are given in the table below:

$x$	1	3	7	10	12	13	16	17	20
$f(x)$	3	6	1	9	15	2	4	5	6

a) Give the middle approximation with 2 subintervals for  $f$  on the interval  $[1, 20]$

b) Use right-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[3, 17]$

c) Use left-handed rectangles to approximate the area with 4 subintervals for  $f$  on the interval  $[1, 12]$

d) Use trapezoids to approximate the area with 3 subintervals for  $f$  on the interval  $[3, 17]$

Non-AP Calculus 4-2, 4-6 Riemann Sums WS: Using Tables of Values

key

- 1) Selected values of a function,  $f$ , are given in the table below:

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

- a) Give the middle approximation with 3 subintervals for  $f$  on the interval  $[0, 20]$

$$\text{Area} \approx 8(2) + 4(7) + 8(6)$$

$$16 + 28 + 48 = 92$$

- b) Use right-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[0, 20]$

$$\text{Area} \approx 8(3) + 4(3) + 8(10)$$

$$24 + 12 + 80 = 116$$

- c) Use left-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[0, 9]$

$$\text{Area} \approx 5(4) + 3(2) + 1(3)$$

$$20 + 6 + 3 = 29$$

- d) Use trapezoids to approximate the area with 2 subintervals for  $f$  on the interval  $[0, 20]$

$$\text{Area} \approx \frac{9}{2}[4+7] + \frac{11}{2}[7+10]$$

$$\frac{w}{2}[h_1+h_2] \quad \frac{9}{2} + \frac{187}{2} = 143$$

- 2) Selected values of a function,  $f$ , are given in the table below:

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

- a) Give the middle approximation with 2 subintervals for  $f$  on the interval  $[1, 20]$

$$\text{Area} \approx 11(1) + 8(4)$$

$$11 + 32 = 43$$

- b) Use right-handed rectangles to approximate the area with 3 subintervals for  $f$  on the interval  $[3, 17]$

$$\text{Area} \approx 7(9) + 3(2) + 4(5)$$

$$63 + 6 + 20 = 89$$

- c) Use left-handed rectangles to approximate the area with 4 subintervals for  $f$  on the interval  $[1, 12]$

$$\text{Area} \approx 2(3) + 4(6) + 3(1) + 2(9)$$

$$6 + 24 + 3 + 18 = 51$$

- d) Use trapezoids to approximate the area with 3 subintervals for  $f$  on the interval  $[3, 17]$

$$\text{Area} \approx \frac{7}{2}[6+9] + \frac{3}{2}[9+2] + \frac{4}{2}[2+5]$$

$$\frac{w}{2}[h_1+h_2] \quad 52.5 + 16.5 + 14 = 83$$

Review 4.2 4.6 Formulas and Definitions:

Summation Formulas:

1)  $\sum_{i=1}^n 1 =$

2)  $\sum_{i=1}^n i =$

3)  $\sum_{i=1}^n i^2 =$

4)  $\sum_{i=1}^n i^3 =$

5) Area of Trapezoid: \_\_\_\_\_

6) Width formula: \_\_\_\_\_

7) Limit Definition of Area under Curve

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Review 4.2 4.6 Formulas and Definitions:

Summation Formulas:

1)  $\sum_{i=1}^n 1 =$

$$\sum_{i=1}^n 1 = n$$

2)  $\sum_{i=1}^n i =$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

3)  $\sum_{i=1}^n i^2 =$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

4)  $\sum_{i=1}^n i^3 =$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

5) Area of Trapezoid: \_\_\_\_\_

$$Area = \frac{w}{2}(h_1 + h_2)$$

6) Width formula: \_\_\_\_\_

$$width = \frac{b-a}{n}$$

7) Limit Definition of Area under Curve

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (width) * f(a + width * i)$$

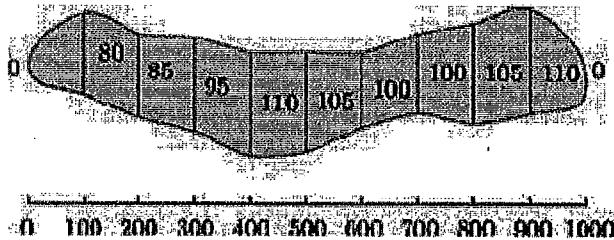
AP Calculus AB 4-2, 4-6 Quiz Review  
Calculators permitted.

Name \_\_\_\_\_

1. Find the sum:  $\sum_{i=2}^4 [(i+1)^2 - (2-i)^3]$

2. Use Sigma notation to write the sum:  $\frac{2}{\sqrt[3]{5-2}} + \frac{4}{\sqrt[3]{5-4}} + \frac{6}{\sqrt[3]{5-6}} + \frac{8}{\sqrt[3]{5-8}}$

3. The width, in feet, at various points along the fairway of a hole on a golf course is given to the right. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway using trapezoids.



4. Use 3 midpoint rectangles to approximate the area of the region bounded by  $f(x) = x^2 + 3$ , the  $x$ -axis,  $x = 0$ , and  $x = 5$ .

5. Use the table of values on the right to estimate the below:

$x$	0	4	6	7	10
$f(x)$	5	3	2	3	5

a. Use 3 left-handed rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on  $[0, 7]$

b. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on  $[0, 10]$

c. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on  $[4, 10]$

d. Use 3 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on  $[0, 7]$

6. Given the region bounded by  $g(x) = 6 - x^2$ , the x-axis,  $x = -1$ , and  $x = 2$ . Use the limit definition to find the exact area of the region.

AP Calculus AB 4-2, 4-6 Quiz Review  
Calculators permitted.

Name Solution Key

1. Find the sum:  $\sum_{i=2}^4 [(i+1)^2 - (2-i)^3]$

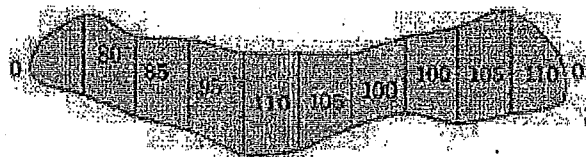
$$(2+1)^2 - (2-2)^3 + (3+1)^2 - (2-3)^3 + (4+1)^2 - (2-4)^3$$

$$9 - 0 + 16 - (-1) + 25 - (-8) = \boxed{59}$$

2. Use Sigma notation to write the sum:  $\frac{2}{\sqrt[3]{5-2}} + \frac{4}{\sqrt[3]{5-4}} + \frac{6}{\sqrt[3]{5-6}} + \frac{8}{\sqrt[3]{5-8}}$

$$\sum_{i=1}^4 \frac{2i}{\sqrt[3]{5-2i}}$$

3. The width, in feet, at various points along the fairway of a hole on a golf course is given to the right. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway using trapezoids.



ft. 100 200 300 400 500 600 700 800 900 1000

$$A = \frac{W}{2} [h_1 + 2h_2 + 2h_3 + 2h_4 + \dots + h_n]$$

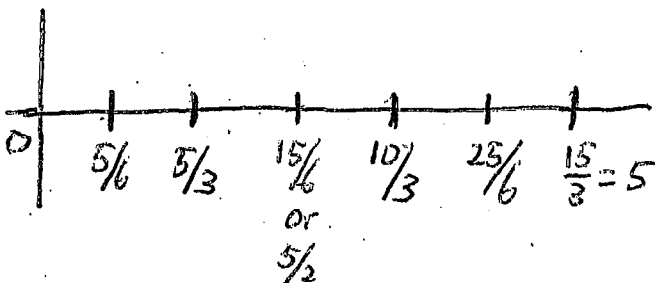
$$A \approx \frac{100}{2} [0 + 2(20) + 2(25) + 2(95) + 2(110) + 2(105) + 2(100) + 2(100) + 2(105) + 2(110) + 0]$$

$$A \approx 50(1780) = 89000 \text{ ft}^2$$

$$\text{fertilizer needed} = 89000 \text{ ft}^2 \cdot \frac{1 \text{ lb fertilizer}}{200 \text{ ft}^2} = \boxed{445 \text{ pounds of fertilizers}}$$

4. Use 3 midpoint rectangles to approximate the area of the region bounded by  $f(x) = x^2 + 3$ , the x-axis,  $x = 0$ , and  $x = 5$ .

$$W = \frac{b-a}{n} = \frac{5-0}{3} = \frac{5}{3}$$



$$A \approx \frac{5}{3} f\left(\frac{5}{6}\right) + \frac{5}{3} f\left(\frac{5}{2}\right) + \frac{5}{3} f\left(\frac{25}{6}\right)$$

$$\approx \frac{5}{3} (33.3) = \boxed{55.5}$$

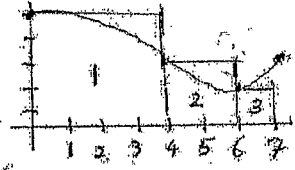
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5. Use the table of values on the right to estimate the below:

$x$	0	4	6	7	10
$f(x)$	5	3	2	3	5

a. Use 3 left-handed rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on  $[0, 7]$

$x$	0	4	6	7
$f(x)$	5	3	2	3



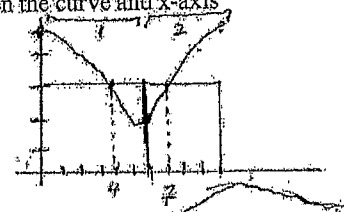
$$A = 4 \cdot f(0) + 2 \cdot f(4) + 1 \cdot f(6)$$

$$= 4(5) + 2(3) + 1(2)$$

$$= \boxed{28}$$

b. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on  $[0, 10]$

$x$	0	4	6	7	10
$f(x)$	5	3	2	3	5



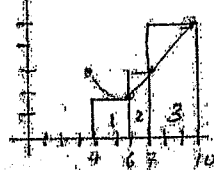
$$A = 6 \cdot f(4) + 4 \cdot f(7)$$

$$= 6(3) + 4(3)$$

$$= \boxed{30}$$

c. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on  $[4, 10]$

$x$	4	6	7	10
$f(x)$	3	2	3	5



$$A = 2f(6) + 1f(7) + 3f(10)$$

$$= 2(2) + 1(3) + 3(5)$$

$$= 4 + 3 + 15 = \boxed{22}$$

d. Use 3 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on  $[0, 7]$

$x$	0	4	6	7
$f(x)$	5	3	2	3

$$A = \frac{w}{2} [h_1 + h_2]$$

$$\frac{1}{2} [f(x) + f(x+1)]$$

$$A = \frac{4}{2} [f(0) + f(4)] + \frac{2}{2} [f(4) + f(6)] + \frac{1}{2} [f(6) + f(7)]$$

$$= 2(5+3) + 1(3+2) + \frac{1}{2}(2+3)$$

$$= 2(8) + 1(5) + \frac{1}{2}(5)$$

$$= 16 + 5 + \frac{5}{2} = \boxed{23.5 \text{ or } \frac{47}{2}}$$

6. Given the region bounded by  $g(x) = 6 - x^2$ , the x-axis,  $x = -1$ , and  $x = 2$ . Use the limit definition to find the exact area of the region.

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{b-a}{n} \right) \cdot f \left[ a + \left( \frac{b-a}{n} \right) i \right]$$

$$A = \lim_{n \rightarrow \infty} \sum \left( \frac{3}{n} \right) \cdot f \left[ -1 + \frac{3}{n} i \right]$$

$$\frac{3}{n} \cdot \left[ 6 - \left( -1 + \frac{3}{n} i \right)^2 \right]$$

$$\frac{3}{n} \left[ 6 - \left( \frac{9}{n^2} i^2 - \frac{6}{n} i + 1 \right) \right]$$

$$\frac{3}{n} \left[ 6 - \frac{9}{n^2} i^2 + \frac{6}{n} i - 1 \right]$$

$$\frac{3}{n} \left[ 5 - \frac{9}{n^2} i^2 + \frac{6}{n} i \right]$$

$$\sum \frac{15}{n} - \sum \frac{27}{n^3} i^2 + \sum \frac{18}{n^2} i$$

$$\frac{15}{n} \sum 1 - \frac{27}{n^3} \sum i^2 + \frac{18}{n^2} \sum i$$

$$\lim_{n \rightarrow \infty} \frac{15}{n} (n) - \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2} \frac{n(n+1)}{2}$$

$$\lim_{n \rightarrow \infty} \frac{15n}{n} - \frac{54n^3 \dots}{6n^3} + \frac{18n^2 \dots}{2n^2}$$

$$15 - \frac{54}{6} + \frac{18}{2} = \boxed{15}$$



4.2/4.6 Quiz Review WS #3

Use sigma notation to write sum (4.2, 4.6 Quiz Review #3)

$$1) \quad 7 \left[ \frac{3}{6} + 4 \right] + 7 \left[ \frac{6}{6} + 8 \right] + 7 \left[ \frac{9}{6} + 12 \right] + \dots + 7 \left[ \frac{18}{6} + 24 \right]$$

2) Use Limit Definition to find area:  $k(x) = 3x - x^2$   $[-1, 2]$ .

3) Use 3 right-handed rectangles to approximate area of  $f(x) = 1 + 3x^2$ , x-axis,  $x=2$ ,  $x=4$

4) Use 2 trapezoids to approximate area  $[3, 20]$

x	2	3	6	9	10	11	13	17	19	20	33
f(x)	8	4	1	5	6	9	3	11	4	17	19

Use sigma notation to write sum Review #3  
Key: WS#3

1)  $7 \left[ \frac{3}{6} + 4 \right] + 7 \left[ \frac{6}{6} + 8 \right] + 7 \left[ \frac{9}{6} + 12 \right] + \dots + 7 \left[ \frac{18}{6} + 24 \right]$

$$\sum_{i=1}^6 7 \left[ \frac{3i}{6} + 4i \right]$$

2) Use Limit Definition to find area:  $h(x) = 3x - x^2$   $[-1, 2]$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{b-a}{n} f \left[ a + \frac{(i-1)(b-a)}{n} \right] \right]$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{3}{n} f \left[ 1 + \frac{3(i-1)}{n} \right] \right]$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{3}{n} \left[ 3 \left( 1 + \frac{3(i-1)}{n} \right) - \left( 1 + \frac{3(i-1)}{n} \right)^2 \right] \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{3}{n} \left[ -3 + \frac{9}{n} i - \left( 1 - \frac{6}{n} i + \frac{9}{n^2} i^2 \right) \right] \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{3}{n} \left[ -4 + \frac{15}{n} i - \frac{9}{n^2} i^2 \right] \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[ \frac{-12}{n} + \frac{45}{n^2} i - \frac{27}{n^3} i^2 \right]$$

$$= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \frac{-12}{n} + \sum_{i=1}^n \frac{45}{n^2} i - \sum_{i=1}^n \frac{27}{n^3} i^2 \right]$$

$$\text{width} = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{-12}{n} \sum_{i=1}^n 1 + \frac{45}{n^2} \sum_{i=1}^n i - \frac{27}{n^3} \sum_{i=1}^n i^2$$

$$= \lim_{n \rightarrow \infty} \frac{-12}{n} (n) + \frac{45}{n^2} \left[ \frac{n(n+1)}{2} \right] - \frac{27}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right]$$

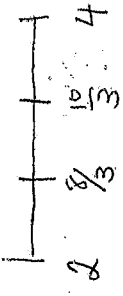
$$= \lim_{n \rightarrow \infty} \frac{-12n}{n} + \frac{45n^2 + 45n}{2n^2} - \frac{54n^3 + \dots}{6n^3}$$

$$= -12 + \frac{45}{2} - \frac{54}{6}$$

$$= \frac{3}{2} \text{ or } 1.5$$

3) Use 3 right-handed rectangles to approximate area of  $f(x) = 1 + 3x^2$ , x-axis,  $x=2$ ,  $x=4$

$$\text{width} = \frac{b-a}{n} = \frac{4-2}{3} = \frac{2}{3}$$



$$A = \frac{2}{3} f\left(\frac{8}{3}\right) + \frac{2}{3} f\left(\frac{10}{3}\right) + \frac{2}{3} f(4)$$

$$A = \frac{2}{3} [22.33 + 34.33 + 49] = \boxed{70.44}$$

4) Use 2 trapezoids to approximate area  $[3, 20]$

x	2	3	6	9	10	11	13	17	19	20	33
f(x)	8	4	1	5	6	9	3	11	4	17	19

$$A = \frac{w}{2} [h_1 + h_2]$$

$$A = \frac{8}{2} [f(3) + f(11)] + \frac{9}{2} [f(11) + f(20)]$$

$$= 4(4+9) + \frac{9}{2}(9+17)$$

$$= 4(13) + 4.5(26)$$

$$= 52 + 117 = \boxed{169}$$

AP Calculus AB 4-2, 4-6 Morning Review  
Calculators permitted.

Name \_\_\_\_\_

1. Find the sum:  $\sum_{i=1}^3 [(2i+1)^2 - (3i+1)^3]$

2. Use Sigma notation to write the sum:  $\frac{7\sqrt{3}}{27} + \frac{7\sqrt{4}}{64} + \frac{7\sqrt{5}}{125} + \frac{7\sqrt{6}}{216} + \dots + \frac{7\sqrt{n}}{n^3}$

3. Use 4 middle rectangles to approximate the area of the region bounded by  $f(x) = 3 + 2x^2$ , the x-axis,  $x = 1$ , and  $x = 7$ .

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [1, 15]

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [5, 11]

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 15]

5. Given the region bounded by  $g(x) = 3 - 2x^2$ , the x-axis,  $x = -1$ , and  $x = 1$ . Use the limit definition to find the exact area of the region.

Calculators permitted.

Name

Key

1. Find the sum:  $\sum_{i=1}^3 [(2i+1)^2 - (3i+1)^3] = (2+1)^2 - (3+1)^3 + (4+1)^2 - (6+1)^3 + (6+1)^2 - (9+1)^3$   
 $= 3^2 - 4^3 + 5^2 - 7^3 + 7^2 - 10^3 = \boxed{-1324}$

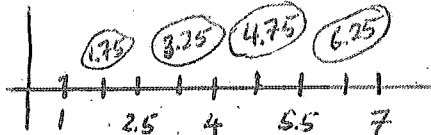
2. Use Sigma notation to write the sum:  $\frac{7\sqrt{3}}{27} + \frac{7\sqrt{4}}{64} + \frac{7\sqrt{5}}{125} + \frac{7\sqrt{6}}{216} + \dots + \frac{7\sqrt{n}}{n^3}$

$$\sum_{i=3}^n \frac{7\sqrt{i}}{i^3}$$

3. Use 4 middle rectangles to approximate the area of the region bounded by

$f(x) = 3+2x^2$ , the  $x$ -axis,  $x = 1$ , and  $x = 7$ .

$w = \frac{b-a}{n} = \frac{7-1}{4} = \frac{3}{2}$



$A \approx (1.5)f(1.75) + 1.5 \cdot f(3.25) + 1.5 \cdot f(4.75) + 1.5 \cdot f(6.25)$   
 $\approx 1.5(9.125 + 24.125 + 48.125 + 81.125)$

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

$\approx (1.5)(16.25)$   
 $\approx \boxed{243.75}$

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and  $x$ -axis on  $[1, 15]$

$5 \cdot f(5) + 5 \cdot f(8) + 4 \cdot f(13)$

$5(2) + 5(3) + 4(6)$

$\boxed{49}$

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and  $x$ -axis on  $[5, 11]$

$1 \cdot f(6) + 2 \cdot f(8) + 3 \cdot f(11)$

$1(7) + 2(3) + 3(1)$

$\boxed{16}$

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and  $x$ -axis on  $[6, 15]$

$A = \frac{W}{2}(h_1 + h_2)$

$= \frac{2}{2}[f(6)+f(8)] + \frac{3}{2}[f(8)+f(11)]$   
 $+ \frac{2}{2}[f(11)+f(13)] + \frac{2}{2}[f(13)+f(15)]$   
 $= \frac{2(10)}{2} + \frac{3(4)}{2} + \frac{2(7)}{2} + \frac{2(11)}{2}$

$\boxed{34}$

5. Given the region bounded by  $g(x) = 3 - 2x^2$ , the  $x$ -axis,  $x = -1$ , and  $x = 1$ . Use the **limit definition** to find the exact area of the region.

$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \text{width} \cdot f[a + \text{width} \cdot i]$

$= \frac{2}{n} \cdot f\left[-1 + \frac{2}{n}i\right]$

$= \frac{2}{n} \cdot \left[3 - 2\left(-1 + \frac{2}{n}i\right)^2\right]$

$= \frac{2}{n} \left[3 - 2\left(1 - \frac{4}{n}i + \frac{4 \cdot 2}{n^2}i^2\right)\right]$

$= \frac{2}{n} \left[3 - 2 + \frac{8}{n}i - \frac{8}{n^2}i^2\right]$

$\sum \frac{2}{n} + \frac{16}{n^2}i - \frac{16 \cdot 2}{n^3}i^2$

$W = \frac{1-(-1)}{n} = \frac{2}{n}$

$\lim_{n \rightarrow \infty} \frac{2}{n} \left[ \sum_{i=1}^n 1 \right] + \frac{16}{n^2} \left[ \sum_{i=1}^n i \right] - \frac{16}{n^3} \left[ \sum_{i=1}^n i^2 \right]$

$\lim_{n \rightarrow \infty} \frac{2}{n}(n) + \frac{16}{n^2} \left[ \frac{n(n+1)}{2} \right] - \frac{16}{n^3} \left[ \frac{n(n+1)(2n+1)}{6} \right]$

$\lim_{n \rightarrow \infty} 2 + \frac{16n^2}{2n^2} - \frac{16(2n^3)}{6n^3}$

$= 2 + 8 - \frac{32}{6}$

$= \boxed{\frac{14}{3}}$