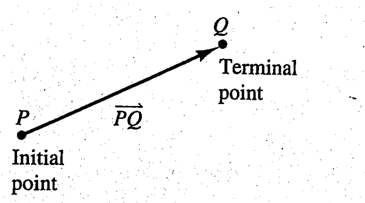
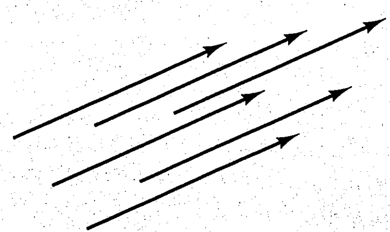
Section 11.1 Vectors in the Plane

- Write the component form of a vector.
- Perform vector operations and interpret the results geometrically.
- Write a vector as a linear combination of standard unit vectors.
- Use vectors to solve problems involving force or velocity.

Component Form of a Vector



A directed line segment
Figure 11.1



Equivalent directed line segments
Figure 11.2

Many quantities in geometry and physics, such as area, volume, temperature, mass, and time, can be characterized by a single real number scaled to appropriate units of measure. These are called **scalar quantities,** and the real number associated with each is called a **scalar.**

Other quantities, such as force, velocity, and acceleration, involve both magnitude and direction and cannot be characterized completely by a single real number. A **directed line segment** is used to represent such a quantity, as shown in Figure 11.1. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** (or **magnitude**) is denoted by $\|\overrightarrow{PQ}\|$. Directed line segments that have the same length and direction are **equivalent,** as shown in Figure 11.2. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector in the plane** and is denoted by $\mathbf{v} = \overrightarrow{PQ}$. In typeset material, vectors are usually denoted by lowercase, boldface letters such as **u**, **v**, and **w**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Be sure you see that a vector in the plane can be represented by many different directed line segments—all pointing in the same direction and all of the same length.

EXAMPLE 1 Vector Representation by Directed Line Segments

Let **v** be represented by the directed line segment from $(0, 0)$ to $(3, 2)$, and let **u** be represented by the directed line segment from $(1, 2)$ to $(4, 4)$. Show that **v** and **u** are equivalent.

Solution Let $P(0, 0)$ and $Q(3, 2)$ be the initial and terminal points of **v**, and let $R(1, 2)$ and $S(4, 4)$ be the initial and terminal points of **u**, as shown in Figure 11.3. You can use the Distance Formula to show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same length.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{PQ}$$
$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13} \qquad \text{Length of } \overrightarrow{RS}$$

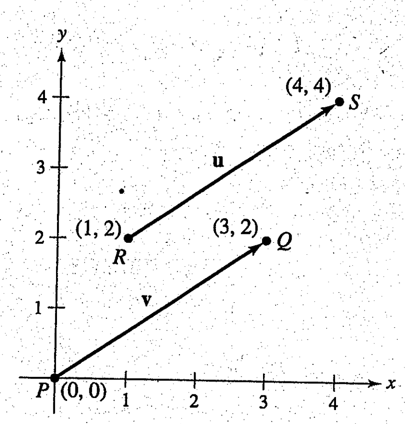
Both line segments have the *same direction,* because they both are directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

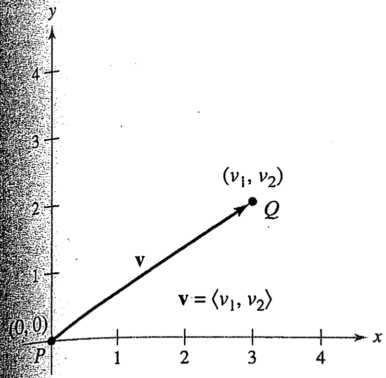
and

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same length and direction, you can conclude that the two vectors are equivalent. That is, **v** and **u** are equivalent.



The vectors **u** and **v** are equivalent.
Figure 11.3



The standard position of a vector
Figure 11.4

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments such as those shown in Figure 11.3. This representation of **v** is said to be in **standard position.** A directed line segment whose initial point is the origin can be uniquely represented by the coordinates of its terminal point $Q(v_1, v_2)$, as shown in Figure 11.4.

Definition of Component Form of a Vector in the Plane

If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is given by

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then **v** is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

This definition implies that two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

The following procedures can be used to convert directed line segments to component form or vice versa.

NOTE It is important to understand that a vector represents a *set* of directed line segments (each having the same length and direction). In practice, however, it is common not to distinguish between a vector and one of its representatives.

1. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, the component form of the vector **v** represented by $\overrightarrow{PQ}$ is $\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$. Moreover, the **length** (or **magnitude**) **of v** is

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} \\ &= \sqrt{v_1^2 + v_2^2}. \end{aligned}$$ Length of a vector

2. If $\mathbf{v} = \langle v_1, v_2 \rangle$, **v** can be represented by the directed line segment, in standard position, from $P(0, 0)$ to $Q(v_1, v_2)$.

The length of **v** is also called the **norm of v**. If $\|\mathbf{v}\| = 1$, **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

EXAMPLE 2 Finding the Component Form and Length of a Vector

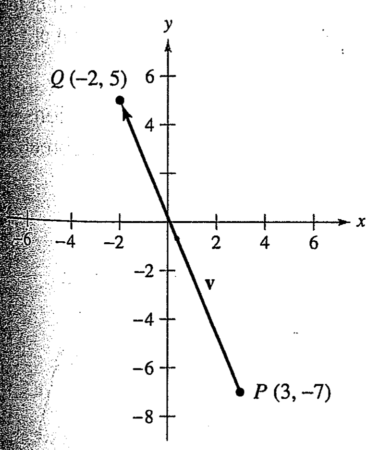
Find the component form and length of the vector **v** that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

Solution Let $P(3, -7) = (p_1, p_2)$ and $Q(-2, 5) = (q_1, q_2)$. Then the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

$$\begin{aligned} v_1 &= q_1 - p_1 = -2 - 3 = -5 \\ v_2 &= q_2 - p_2 = 5 - (-7) = 12. \end{aligned}$$

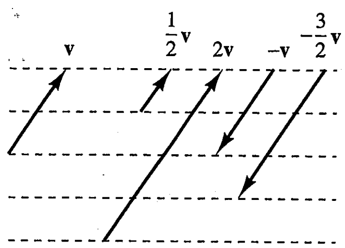
So, as shown in Figure 11.5, $\mathbf{v} = \langle -5, 12 \rangle$, and the length of **v** is

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(-5)^2 + 12^2} \\ &= \sqrt{169} \\ &= 13. \end{aligned}$$



Component form of **v**: $\mathbf{v} = \langle -5, 12 \rangle$
Figure 11.5

Vector Operations



The scalar multiplication of **v**
Figure 11.6

Definitions of Vector Addition and Scalar Multiplication

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let c be a scalar.

1. The **vector sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
2. The **scalar multiple** of c and $\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2 \rangle$.
3. The **negative** of $\mathbf{v}$ is the vector

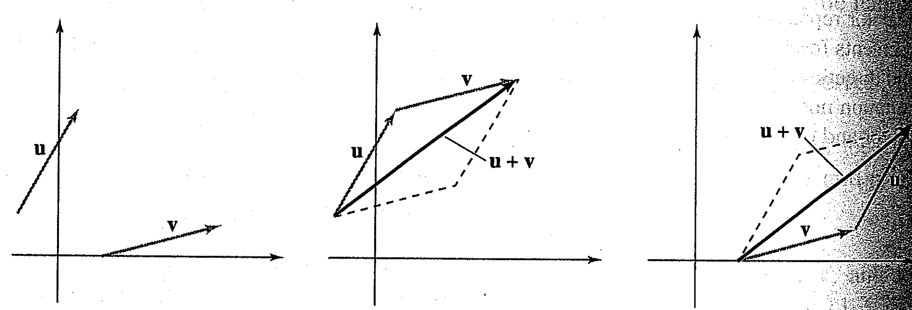
$$-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle.$$

4. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle.$$

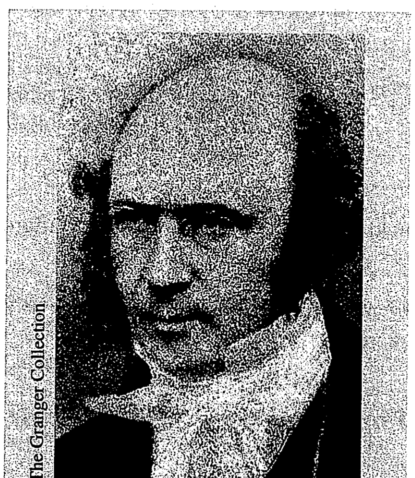
Geometrically, the scalar multiple of a vector $\mathbf{v}$ and a scalar c is the vector that is $|c|$ times as long as $\mathbf{v}$, as shown in Figure 11.6. If c is positive, $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If c is negative, $c\mathbf{v}$ has the opposite direction.

The sum of two vectors can be represented geometrically by positioning the vectors (without changing their magnitudes or directions) so that the initial point of one coincides with the terminal point of the other, as shown in Figure 11.7. The vector $\mathbf{u} + \mathbf{v}$, called the **resultant vector,** is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.



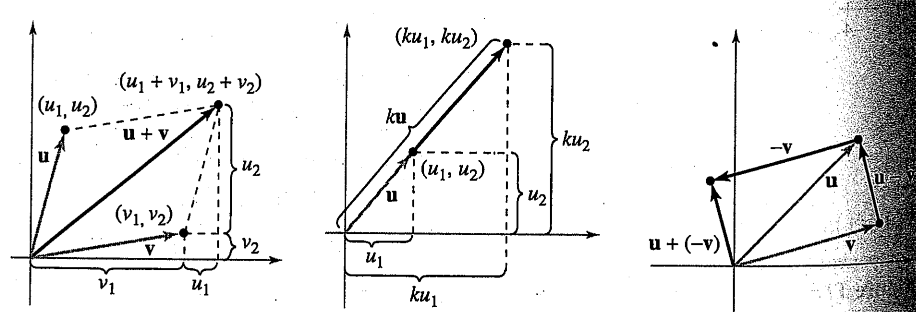
To find $\mathbf{u} + \mathbf{v}$, (1) move the initial point of **v** to the terminal point of **u**, or (2) move the initial point of **u** to the terminal point of **v**.

Figure 11.7



Isaac William Rowan Hamilton (1805–1865)
Some of the earliest work with vectors was done by the Irish mathematician William Rowan Hamilton. Hamilton spent many years developing a system of vector-like quantities called *quaternions*. Although Hamilton was convinced of the benefits of quaternions, the operations he defined did not produce good models for physical phenomena. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.

Figure 11.8 shows the equivalence of the geometric and algebraic definitions of vector addition and scalar multiplication, and presents (at far right) a geometric interpretation of $\mathbf{u} - \mathbf{v}$.



Vector addition
Figure 11.8

Scalar multiplication

Vector subtraction

EXAMPLE 3 Vector Operations

Given $\mathbf{v} = \langle -2, 5\rangle$ and $\mathbf{w} = \langle 3, 4\rangle$, find each of the vectors.

a. $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. $\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}(-2), \frac{1}{2}(5)\right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$

b. $\mathbf{w} - \mathbf{v} = \langle w_1 - v_1, w_2 - v_2\rangle = \langle 3 - (-2), 4 - 5\rangle = \langle 5, -1\rangle$

c. Using $2\mathbf{w} = \langle 6, 8\rangle$, you have

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + \langle 6, 8\rangle \\ &= \langle -2 + 6, 5 + 8\rangle \\ &= \langle 4, 13\rangle.\end{aligned}$$

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the following theorem.

THEOREM 11.1 Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$	
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive Property
8. $1(\mathbf{u}) = \mathbf{u}, 0(\mathbf{u}) = \mathbf{0}$	

Proof The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [\langle u_1, u_2\rangle + \langle v_1, v_2\rangle] + \langle w_1, w_2\rangle \\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle \\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle = \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

Similarly, the proof of the *Distributive Property* of vectors depends on the Distributive Property of real numbers.

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)\langle u_1, u_2\rangle \\ &= \langle (c + d)u_1, (c + d)u_2\rangle \\ &= \langle cu_1 + du_1, cu_2 + du_2\rangle \\ &= \langle cu_1, cu_2\rangle + \langle du_1, du_2\rangle = c\mathbf{u} + d\mathbf{u}\end{aligned}$$

The other properties can be proved in a similar manner.

The Granger Collection

EMMY NOETHER (1882–1935)
One person who contributed to our knowledge of axiomatic systems was the German mathematician Emmy Noether. Noether is generally recognized as the leading woman mathematician in recent history.

FOR FURTHER INFORMATION For more information on Emmy Noether, see the article "Emmy Noether, Greatest Woman Mathematician" by Clark Kimberling in *The Mathematics Teacher.* To view this article, go to the website *www.matharticles.com.*

Any set of vectors (with an accompanying set of scalars) that satisfies the eight properties given in Theorem 11.1 is a **vector space.*** The eight properties are the *vector space axioms.* So, this theorem states that the set of vectors in the plane (with the set of real numbers) forms a vector space.

THEOREM 11.2 Length of a Scalar Multiple

Let $\mathbf{v}$ be a vector and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|. \qquad |c| \text{ is the absolute value of } c.$$

Proof Because $c\mathbf{v} = \langle cv_1, cv_2\rangle$, it follows that

$$\begin{aligned}\|c\mathbf{v}\| = \|\langle cv_1, cv_2\rangle\| &= \sqrt{(cv_1)^2 + (cv_2)^2}\\ &= \sqrt{c^2v_1^2 + c^2v_2^2}\\ &= \sqrt{c^2(v_1^2 + v_2^2)}\\ &= |c|\sqrt{v_1^2 + v_2^2}\\ &= |c|\,\|\mathbf{v}\|.\end{aligned}$$

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given vector. The following theorem gives a procedure for doing this.

THEOREM 11.3 Unit Vector in the Direction of v

If $\mathbf{v}$ is a nonzero vector in the plane, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has length 1 and the same direction as $\mathbf{v}$.

Proof Because $1/\|\mathbf{v}\|$ is positive and $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, you can conclude that $\mathbf{u}$ has the same direction as $\mathbf{v}$. To see that $\|\mathbf{u}\| = 1$, note that

$$\begin{aligned}\|\mathbf{u}\| &= \left\|\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\right\|\\ &= \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\|\\ &= \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\|\\ &= 1.\end{aligned}$$

So, $\mathbf{u}$ has length 1 and the same direction as $\mathbf{v}$.

In Theorem 11.3, $\mathbf{u}$ is called a **unit vector in the direction of v.** The process of multiplying $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to get a unit vector is called **normalization of v.**

* *For more information about vector spaces, see* Elementary Linear Algebra, *Fifth Edition, by Larson, Edwards, and Falvo (Boston: Houghton Mifflin Company, 2004).*

EXAMPLE 4 Finding a Unit Vector

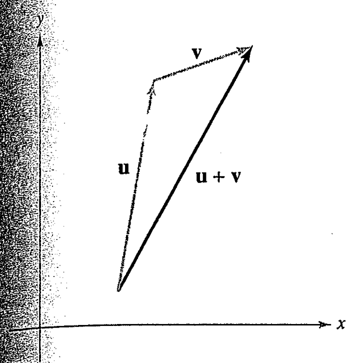
Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that it has length 1.

Solution From Theorem 11.3, the unit vector in the direction of **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}}\langle -2, 5\rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.$$

This vector has length 1, because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$



Triangle inequality
Figure 11.9

Generally, the length of the sum of two vectors is not equal to the sum of their lengths. To see this, consider the vectors **u** and **v** as shown in Figure 11.9. By considering **u** and **v** as two sides of a triangle, you can see that the length of the third side is $\|\mathbf{u} + \mathbf{v}\|$, and you have

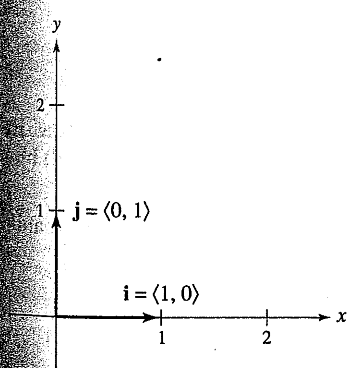
$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Equality occurs only if the vectors **u** and **v** have the *same direction.* This result is called the **triangle inequality** for vectors. (You are asked to prove this in Exercise 89, Section 11.3.)

Standard Unit Vectors

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** in the plane and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle \qquad \text{Standard unit vectors}$$



Standard unit vectors **i** and **j**
Figure 11.10

as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of **i** and **j**. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**

EXAMPLE 5 Writing a Linear Combination of Unit Vectors

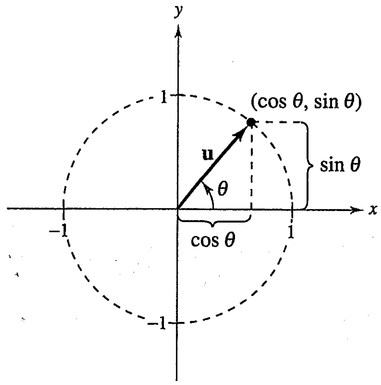
Let **u** be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$, and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of **i** and **j**.

a. **u** **b.** $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$

Solution

a. $\mathbf{u} = \langle q_1 - p_1, q_2 - p_2\rangle$
$= \langle -1 - 2, 3 - (-5)\rangle$
$= \langle -3, 8\rangle = -3\mathbf{i} + 8\mathbf{j}$

b. $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j})$
$= -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j}$
$= -12\mathbf{i} + 19\mathbf{j}$



The angle θ from the positive x-axis to the vector **u**
Figure 11.11

If **u** is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to **u**, then the terminal point of **u** lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \qquad \text{Unit vector}$$

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector **v** making an angle θ with the positive x-axis has the same direction as **u**, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j}.$$

EXAMPLE 6 Writing a Vector of Given Magnitude and Direction

The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

Solution Because the angle between **v** and the positive x-axis is $\theta = \pi/6$, you can write the following.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j} \\ &= 3\cos\frac{\pi}{6}\mathbf{i} + 3\sin\frac{\pi}{6}\mathbf{j} \\ &= \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\end{aligned}$$

Applications of Vectors

Vectors have many applications in physics and engineering. One example is force. A vector can be used to represent force because force has both magnitude and direction. If two or more forces are acting on an object, then the **resultant force** on the object is the vector sum of the vector forces.

EXAMPLE 7 Finding the Resultant Force

Two tugboats are pushing an ocean liner, as shown in Figure 11.12. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner?

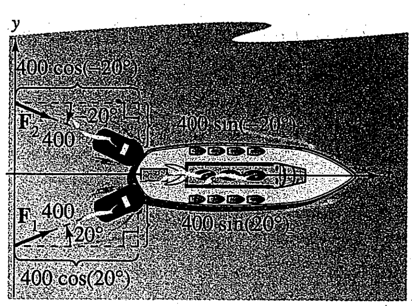
Solution Using Figure 11.12, you can represent the forces exerted by the first and second tugboats as

$$\begin{aligned}\mathbf{F}_1 &= 400\langle \cos 20°, \sin 20° \rangle \\ &= 400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j} \\ \mathbf{F}_2 &= 400\langle \cos(-20°), \sin(-20°) \rangle \\ &= 400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}.\end{aligned}$$

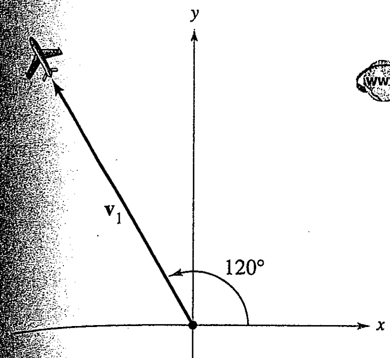
The resultant force on the ocean liner is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= [400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}] + [400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}] \\ &= 800\cos(20°)\mathbf{i} \\ &\approx 752\mathbf{i}.\end{aligned}$$

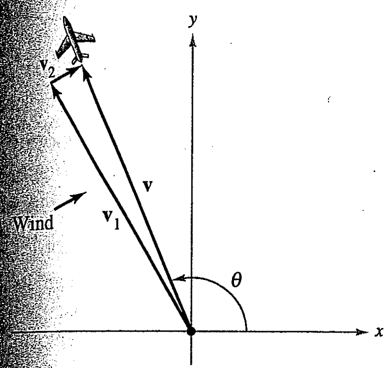
So, the resultant force on the ocean liner is approximately 752 pounds in the direction of the positive x-axis.



The resultant force on the ocean liner that is exerted by the two tugboats.
Figure 11.12



(a) Direction without wind



(b) Direction with wind
Figure 11.13

In surveying and navigation, a bearing is a direction that measures the acute angle that a path or line of sight makes with a fixed north-south line. In air navigation, bearings are measured in degrees clockwise from north.

EXAMPLE 8 Finding a Velocity

An airplane is traveling at a fixed altitude with a negligible wind factor. The airplane is traveling at a speed of 500 miles per hour with a bearing of 330°, as shown in Figure 11.13(a). As the airplane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction N 45° E (45° east of north), as shown in Figure 11.13(b). What are the resultant speed and direction of the airplane?

Solution Using Figure 11.13(a), represent the velocity of the airplane (alone) as

$$\mathbf{v}_1 = 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}.$$

The resultant velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 &= 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j} + 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

To find the resultant speed and direction, write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Because $\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$, you can write

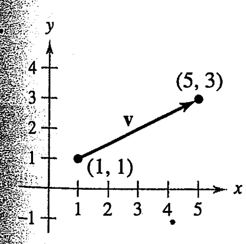
$$\mathbf{v} \approx 522.5\left(\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right) \approx 522.5,[\cos(112.6°)\mathbf{i} + \sin(112.6°)\mathbf{j}].$$

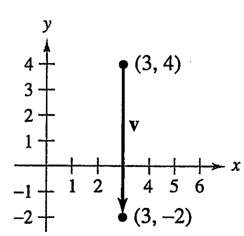
The new speed of the airplane, as altered by the wind, is approximately 522.5 miles per hour in a path that makes an angle of 112.6° with the positive x-axis.

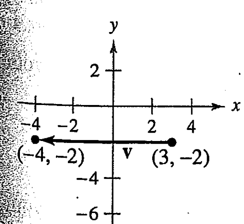
Exercises for Section 11.1

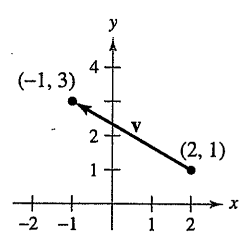
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.

1.


2.


3.


4.


In Exercises 5–8, find the vectors u and v whose initial and terminal points are given. Show that u and v are equivalent.

5. **u**: (3, 2), (5, 6)
   v: (−1, 4), (1, 8)
6. **u**: (−4, 0), (1, 8)
   v: (2, −1), (7, 7)
7. **u**: (0, 3), (6, −2)
   v: (3, 10), (9, 5)
8. **u**: (−4, −1), (11, −4)
   v: (10, 13), (25, 10)

In Exercises 9–16, the initial and terminal points of a vector v are given. (a) Sketch the given directed line segment, (b) write the vector in component form, and (c) sketch the vector with its initial point at the origin.

	Initial Point	*Terminal Point*		*Initial Point*	*Terminal Point*
9.	(1, 2)	(5, 5)	**10.**	(2, −6)	(3, 6)
11.	(10, 2)	(6, −1)	**12.**	(0, −4)	(−5, −1)

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

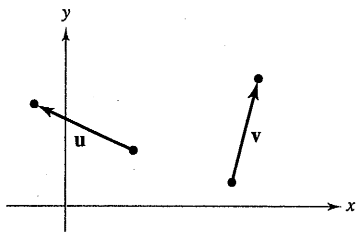
	Initial Point	*Terminal Point*		*Initial Point*	*Terminal Point*
13.	$(6, 2)$	$(6, 6)$	**14.**	$(7, -1)$	$(-3, -1)$
15.	$\left(\frac{3}{2}, \frac{4}{3}\right)$	$\left(\frac{1}{2}, 3\right)$	**16.**	$(0.12, 0.60)$	$(0.84, 1.25)$

In Exercises 17 and 18, sketch each scalar multiple of v.

17. $\mathbf{v} = \langle 2, 3\rangle$
(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$ (c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$

18. $\mathbf{v} = \langle -1, 5\rangle$
(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$ (c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

In Exercises 19–22, use the figure to sketch a graph of the vector. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



19. $-\mathbf{u}$ **20.** $2\mathbf{u}$
21. $\mathbf{u} - \mathbf{v}$ **22.** $\mathbf{u} + 2\mathbf{v}$

In Exercises 23 and 24, find (a) $\frac{2}{3}\mathbf{u}$, (b) $\mathbf{v} - \mathbf{u}$, and (c) $2\mathbf{u} + 5\mathbf{v}$.

23. $\mathbf{u} = \langle 4, 9\rangle$, $\mathbf{v} = \langle 2, -5\rangle$
24. $\mathbf{u} = \langle -3, -8\rangle$, $\mathbf{v} = \langle 8, 25\rangle$

In Exercises 25–28, find the vector v where $\mathbf{u} = \langle 2, -1\rangle$ and $\mathbf{w} = \langle 1, 2\rangle$. Illustrate the vector operations geometrically.

25. $\mathbf{v} = \frac{3}{2}\mathbf{u}$ **26.** $\mathbf{v} = \mathbf{u} + \mathbf{w}$
27. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$ **28.** $\mathbf{v} = 5\mathbf{u} - 3\mathbf{w}$

In Exercises 29 and 30, the vector v and its initial point are given. Find the terminal point.

29. $\mathbf{v} = \langle -1, 3\rangle$; Initial point: $(4, 2)$
30. $\mathbf{v} = \langle 4, -9\rangle$; Initial point: $(3, 2)$

In Exercises 31–36, find the magnitude of v.

31. $\mathbf{v} = \langle 4, 3\rangle$ **32.** $\mathbf{v} = \langle 12, -5\rangle$
33. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$ **34.** $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$
35. $\mathbf{v} = 4\mathbf{j}$ **36.** $\mathbf{v} = \mathbf{i} - \mathbf{j}$

In Exercises 37–40, find the unit vector in the direction of u and verify that it has length 1.

37. $\mathbf{u} = \langle 3, 12\rangle$ **38.** $\mathbf{u} = \langle 5, 15\rangle$
39. $\mathbf{u} = \left\langle \frac{3}{2}, \frac{5}{2}\right\rangle$ **40.** $\mathbf{u} = \langle -6.2, 3.4\rangle$

In Exercises 41–44, find the following.

(a) $\|\mathbf{u}\|$ **(b)** $\|\mathbf{v}\|$ **(c)** $\|\mathbf{u} + \mathbf{v}\|$

(d) $\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$ **(e)** $\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\|$ **(f)** $\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\|$

41. $\mathbf{u} = \langle 1, -1\rangle$, $\mathbf{v} = \langle -1, 2\rangle$
42. $\mathbf{u} = \langle 0, 1\rangle$, $\mathbf{v} = \langle 3, -3\rangle$
43. $\mathbf{u} = \left\langle 1, \frac{1}{2}\right\rangle$, $\mathbf{v} = \langle 2, 3\rangle$
44. $\mathbf{u} = \langle 2, -4\rangle$, $\mathbf{v} = \langle 5, 5\rangle$

In Exercises 45 and 46, sketch a graph of u, v, and u + v. Then demonstrate the triangle inequality using the vectors u and v.

45. $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle 5, 4\rangle$ **46.** $\mathbf{u} = \langle -3, 2\rangle$, $\mathbf{v} = \langle 1, -2\rangle$

In Exercises 47–50, find the vector v with the given magnitude and the same direction as u.

	Magnitude	*Direction*
47.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1\rangle$
48.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle -1, 1\rangle$
49.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \langle \sqrt{3}, 3\rangle$
50.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 0, 3\rangle$

In Exercises 51–54, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis.

51. $\|\mathbf{v}\| = 3, \quad \theta = 0°$ **52.** $\|\mathbf{v}\| = 5, \quad \theta = 120°$
53. $\|\mathbf{v}\| = 2, \quad \theta = 150°$ **54.** $\|\mathbf{v}\| = 1, \quad \theta = 3.5°$

In Exercises 55–58, find the component form of u + v given the lengths of u and v and the angles that u and v make with the positive *x*-axis.

55. $\|\mathbf{u}\| = 1, \quad \theta_{\mathbf{u}} = 0°$; $\|\mathbf{v}\| = 3, \quad \theta_{\mathbf{v}} = 45°$
56. $\|\mathbf{u}\| = 4, \quad \theta_{\mathbf{u}} = 0°$; $\|\mathbf{v}\| = 2, \quad \theta_{\mathbf{v}} = 60°$
57. $\|\mathbf{u}\| = 2, \quad \theta_{\mathbf{u}} = 4$; $\|\mathbf{v}\| = 1, \quad \theta_{\mathbf{v}} = 2$
58. $\|\mathbf{u}\| = 5, \quad \theta_{\mathbf{u}} = -0.5$; $\|\mathbf{v}\| = 5, \quad \theta_{\mathbf{v}} = 0.5$

Writing About Concepts

59. In your own words, state the difference between a scalar and a vector. Give examples of each.

60. Give geometric descriptions of the operations of addition of vectors and multiplication of a vector by a scalar.

61. Identify the quantity as a scalar or as a vector. Explain your reasoning.
(a) The muzzle velocity of a gun
(b) The price of a company's stock

62. Identify the quantity as a scalar or as a vector. Explain your reasoning.
(a) The air temperature in a room
(b) The weight of a car

In Exercises 63–68, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{w} = \langle 1, -1\rangle$.

63. $\mathbf{v} = \langle 2, 1\rangle$
64. $\mathbf{v} = \langle 0, 3\rangle$
65. $\mathbf{v} = \langle 3, 0\rangle$
66. $\mathbf{v} = \langle 3, 3\rangle$
67. $\mathbf{v} = \langle 1, 1\rangle$
68. $\mathbf{v} = \langle -1, 7\rangle$

In Exercises 69–74, find a unit vector (a) parallel to and (b) normal to the graph of $f(x)$ at the given point. Then sketch a graph of the vectors and the function.

	Function	*Point*
69.	$f(x) = x^2$	$(3, 9)$
70.	$f(x) = -x^2 + 5$	$(1, 4)$
71.	$f(x) = x^3$	$(1, 1)$
72.	$f(x) = x^3$	$(-2, -8)$
73.	$f(x) = \sqrt{25 - x^2}$	$(3, 4)$
74.	$f(x) = \tan x$	$\left(\frac{\pi}{4}, 1\right)$

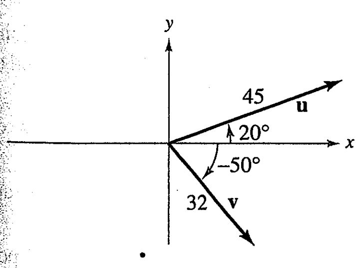
In Exercises 75 and 76, find the component form of $\mathbf{v}$ given the magnitudes of $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ and the angles that $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$ make with the positive x-axis.

75. $\|\mathbf{u}\| = 1, \theta = 45°$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}, \theta = 90°$

76. $\|\mathbf{u}\| = 4, \theta = 30°$
$\|\mathbf{u} + \mathbf{v}\| = 6, \theta = 120°$

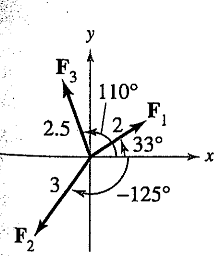
77. ***Programming*** You are given the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ and the angles $\mathbf{u}$ and $\mathbf{v}$ make with the positive x-axis. Write a program for a graphing utility in which the output is the following.
(a) $\mathbf{u} + \mathbf{v}$ (b) $\|\mathbf{u} + \mathbf{v}\|$
(c) The angle $\mathbf{u} + \mathbf{v}$ makes with the positive x-axis

78. ***Programming*** Use the program you wrote in Exercise 77 to find the magnitude and direction of the resultant of the vectors shown.

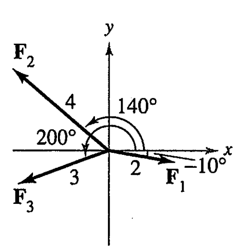


In Exercises 79 and 80, use a graphing utility to find the magnitude and direction of the resultant of the vectors.

79.



80.



81. ***Numerical and Graphical Analysis*** Forces with magnitudes of 180 newtons and 275 newtons act on a hook (see figure). The angle between the two forces is θ degrees.
(a) If $\theta = 30°$, find the direction and magnitude of the resultant force.
(b) Write the magnitude M and direction α of the resultant force as functions of θ, where $0° \le \theta \le 180°$.
(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions M and α.
(e) Explain why one of the functions decreases for increasing values of θ whereas the other does not.

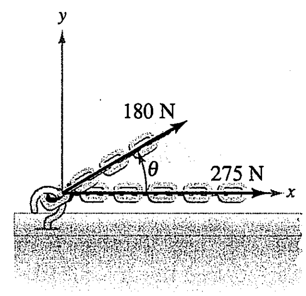


Figure for 81

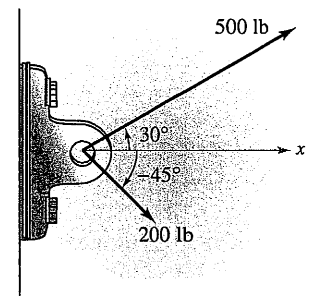


Figure for 82

82. ***Resultant Force*** Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30° and −45°, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant force.

83. ***Resultant Force*** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

84. ***Resultant Force*** Three forces with magnitudes of 400 newtons, 280 newtons, and 350 newtons act on an object at angles of −30°, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

85. ***Think About It*** Consider two forces of equal magnitude acting on a point.
(a) If the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.
(b) If the resultant of the forces is $\mathbf{0}$, make a conjecture about the angle between the forces.
(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

86. ***Graphical Reasoning*** Consider two forces $\mathbf{F}_1 = \langle 20, 0\rangle$ and $\mathbf{F}_2 = 10\langle \cos\theta, \sin\theta\rangle$.

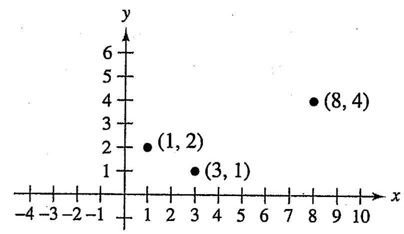
(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$.

(b) Determine the magnitude of the resultant as a function of θ. Use a graphing utility to graph the function for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum and for what value of θ does it occur? What is its minimum and for what value of θ does it occur?

(d) Explain why the magnitude of the resultant is never 0.

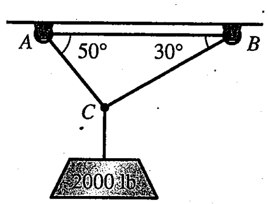
87. Three vertices of a parallelogram are $(1, 2)$, $(3, 1)$, and $(8, 4)$. Find the three possible fourth vertices (see figure).



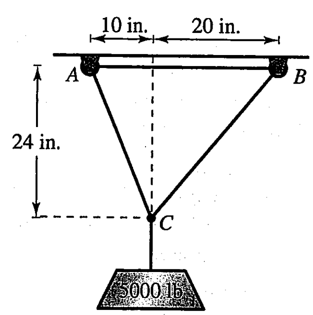
88. Use vectors to find the points of trisection of the line segment with endpoints $(1, 2)$ and $(7, 5)$.

Cable Tension **In Exercises 89 and 90, use the figure to determine the tension in each cable supporting the given load.**

89.



90.



91. ***Projectile Motion*** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

92. ***Shared Load*** To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).

(a) Find each rope's tension if the resultant force is vertical.

(b) Find the vertical component of each worker's force.

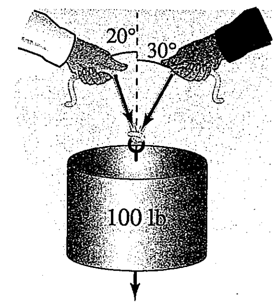


Figure for 92

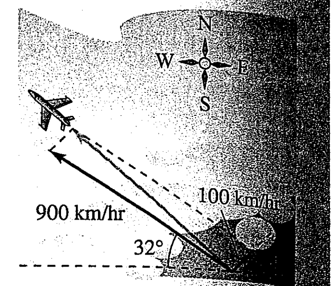


Figure for 93

93. ***Navigation*** A plane is flying in the direction 302°. Its speed with respect to the air is 900 kilometers per hour. The wind at the plane's altitude is from the southwest at 100 kilometers per hour (see figure). What is the true direction of the plane, and what is its speed with respect to the ground?

94. ***Navigation*** A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-mile-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

True or False? **In Exercises 95–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude and direction, then $\mathbf{u}$ and $\mathbf{v}$ are equivalent.

96. If $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$, then $\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$.

97. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

98. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

99. If $a = b$, then $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}a$.

100. If $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude but opposite directions, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

101. Prove that $\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}$ and $\mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ are unit vectors for any angle θ.

102. ***Geometry*** Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to, and one-half the length of, the third side.

103. ***Geometry*** Using vectors, prove that the diagonals of a parallelogram bisect each other.

104. Prove that the vector $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

105. Consider the vector $\mathbf{u} = \langle x, y\rangle$. Describe the set of all points (x, y) such that $\|\mathbf{u}\| = 5$.

Putnam Exam Challenge

106. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant $(= v_0)$, determine the set H of points in the plane and above the horizontal which can be hit.

This problem was composed by the Committee on the Putnam Prize Competition.