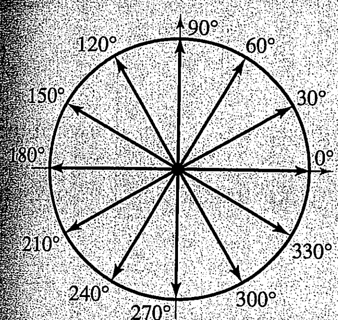
Section 11.3 The Dot Product of Two Vectors

- **Use properties of the dot product of two vectors.**
- **Find the angle between two vectors using the dot product.**
- **Find the direction cosines of a vector in space.**
- **Find the projection of a vector onto another vector.**
- **Use vectors to find the work done by a constant force.**

EXPLORATION

Interpreting a Dot Product Several vectors are shown below on the unit circle. Find the dot products of several pairs of vectors. Then find the angle between each pair that you used. Make a conjecture about the relationship between the dot product of two vectors and the angle between the vectors.



The Dot Product

So far you have studied two operations with vectors—vector addition and multiplication by a scalar—each of which yields another vector. In this section you will study a third vector operation, called the **dot product.** This product yields a scalar, rather than a vector.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

NOTE Because the dot product of two vectors yields a scalar, it is also called the **inner product** (or **scalar product**) of the two vectors.

THEOREM 11.4 Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative Property
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive Property
3. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{v} = 0$
5. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Proof To prove the first property, let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= v_1u_1 + v_2u_2 + v_3u_3 \\ &= \mathbf{v} \cdot \mathbf{u}.\end{aligned}$$

For the fifth property, let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\mathbf{v} \cdot \mathbf{v} &= v_1^2 + v_2^2 + v_3^2 \\ &= \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 \\ &= \|\mathbf{v}\|^2.\end{aligned}$$

Proofs of the other properties are left to you.

EXAMPLE 1 Finding Dot Products

Given $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 5, 8 \rangle$, and $\mathbf{w} = \langle -4, 3 \rangle$, find each of the following.

a. $\mathbf{u} \cdot \mathbf{v}$ **b.** $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

c. $\mathbf{u} \cdot (2\mathbf{v})$ **d.** $\|\mathbf{w}\|^2$

Solution

a. $\mathbf{u} \cdot \mathbf{v} = \langle 2, -2 \rangle \cdot \langle 5, 8 \rangle = 2(5) + (-2)(8) = -6$

b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\langle -4, 3 \rangle = \langle 24, -18 \rangle$

c. $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$ Theorem 11.4

d. $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ Theorem 11.4

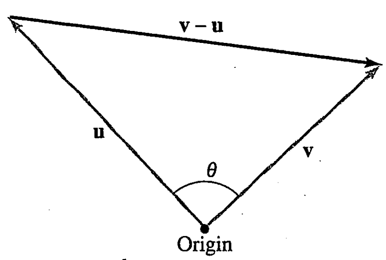
$= \langle -4, 3 \rangle \cdot \langle -4, 3 \rangle$ Substitute $\langle -4, 3 \rangle$ for $\mathbf{w}$.

$= (-4)(-4) + (3)(3)$ Definition of dot product

$= 25$ Simplify.

Notice that the result of part (b) is a *vector* quantity, whereas the results of the other three parts are *scalar* quantities.

Angle Between Two Vectors



The angle between two vectors
Figure 11.24

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 11.24. The next theorem shows how to find this angle using the dot product. (Note that the angle between the zero vector and another vector is not defined here.)

THEOREM 11.5 Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

Proof Consider the triangle determined by vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} - \mathbf{u}$, as shown in Figure 11.24. By the Law of Cosines, you can write

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

Using the properties of the dot product, the left side can be rewritten as

$$\begin{aligned} \|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\ &= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\ &= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 \end{aligned}$$

and substitution back into the Law of Cosines yields

$$\begin{aligned} \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\ -2\mathbf{u} \cdot \mathbf{v} &= -2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\ \cos\theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}. \end{aligned}$$

If the angle between two vectors is known, rewriting Theorem 11.5 in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \qquad \text{Alternative form of dot product}$$

produces an alternative way to calculate the dot product. From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign. Figure 11.25 shows the possible orientations of two vectors.

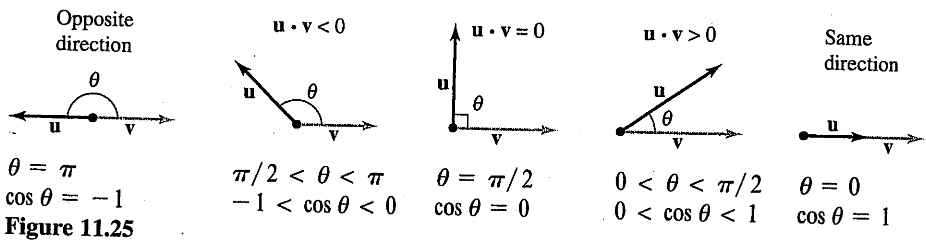


Figure 11.25

From Theorem 11.5, you can see that two nonzero vectors meet at a right angle if and only if their dot product is zero. Two such vectors are said to be **orthogonal.**

> **Definition of Orthogonal Vectors**
>
> The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

NOTE The terms “perpendicular,” “orthogonal,” and “normal” all mean essentially the same thing—meeting at right angles. However, it is common to say that two vectors are *orthogonal*, two lines or planes are *perpendicular*, and a vector is *normal* to a given line or plane.

From this definition, it follows that the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u} = 0$. Moreover, for $0 \le \theta \le \pi$, you know that $\cos \theta = 0$ if and only if $\theta = \pi/2$. So, you can use Theorem 11.5 to conclude that two *nonzero* vectors are orthogonal if and only if the angle between them is $\pi/2$.

EXAMPLE 2 Finding the Angle Between Two Vectors

For $\mathbf{u} = \langle 3, -1, 2\rangle$, $\mathbf{v} = \langle -4, 0, 2\rangle$, $\mathbf{w} = \langle 1, -1, -2\rangle$, and $\mathbf{z} = \langle 2, 0, -1\rangle$, find the angle between each pair of vectors.

a. $\mathbf{u}$ and $\mathbf{v}$ **b.** $\mathbf{u}$ and $\mathbf{w}$ **c.** $\mathbf{v}$ and $\mathbf{z}$

Solution

a. $\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \dfrac{-12 + 0 + 4}{\sqrt{14}\sqrt{20}} = \dfrac{-8}{2\sqrt{14}\sqrt{5}} = \dfrac{-4}{\sqrt{70}}$

Because $\mathbf{u} \cdot \mathbf{v} < 0$, $\theta = \arccos \dfrac{-4}{\sqrt{70}} \approx 2.069$ radians.

b. $\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\| \|\mathbf{w}\|} = \dfrac{3 + 1 - 4}{\sqrt{14}\sqrt{6}} = \dfrac{0}{\sqrt{84}} = 0$

Because $\mathbf{u} \cdot \mathbf{w} = 0$, $\mathbf{u}$ and $\mathbf{w}$ are *orthogonal.* So, $\theta = \pi/2$.

c. $\cos \theta = \dfrac{\mathbf{v} \cdot \mathbf{z}}{\|\mathbf{v}\| \|\mathbf{z}\|} = \dfrac{-8 + 0 - 2}{\sqrt{20}\sqrt{5}} = \dfrac{-10}{\sqrt{100}} = -1$

Consequently, $\theta = \pi$. Note that $\mathbf{v}$ and $\mathbf{z}$ are parallel, with $\mathbf{v} = -2\mathbf{z}$.

Direction Cosines

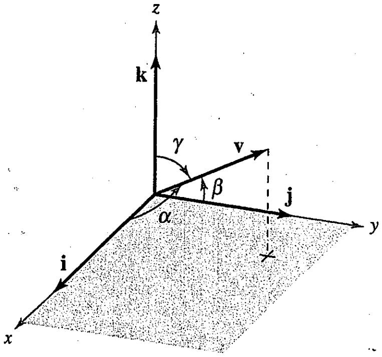
For a vector in the plane, you have seen that it is convenient to measure direction in terms of the angle, measured counterclockwise, *from* the positive x-axis *to* the vector. In space it is more convenient to measure direction in terms of the angles *between* the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 11.26. The angles α, β, and γ are the **direction angles of v,** and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines of v.** Because

$$\mathbf{v}\cdot\mathbf{i} = \|\mathbf{v}\|\,\|\mathbf{i}\|\cos\alpha = \|\mathbf{v}\|\cos\alpha$$

and

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$

it follows that $\cos\alpha = v_1/\|\mathbf{v}\|$. By similar reasoning with the unit vectors $\mathbf{j}$ and $\mathbf{k}$, you have



Direction angles
Figure 11.26

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} \qquad \alpha \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{i}.$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} \qquad \beta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{j}.$$

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}. \qquad \gamma \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{k}.$$

Consequently, any nonzero vector $\mathbf{v}$ in space has the normalized form

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

and because $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 3 Finding Direction Angles

Find the direction cosines and angles for the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution Because $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, you can write the following.

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}} \Rightarrow \alpha \approx 68.2^\circ \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{i}$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{29}} \Rightarrow \beta \approx 56.1^\circ \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{j}$$

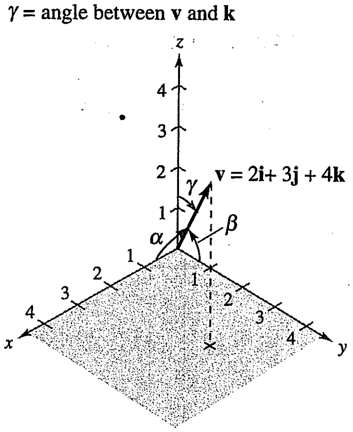
$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{\sqrt{29}} \Rightarrow \gamma \approx 42.0^\circ \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{k}$$

Furthermore, the sum of the squares of the direction cosines is

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{4}{29} + \frac{9}{29} + \frac{16}{29}\\ &= \frac{29}{29}\\ &= 1.\end{aligned}$$

See Figure 11.27.

α = angle between $\mathbf{v}$ and $\mathbf{i}$
β = angle between $\mathbf{v}$ and $\mathbf{j}$
γ = angle between $\mathbf{v}$ and $\mathbf{k}$



The direction angles of $\mathbf{v}$
Figure 11.27

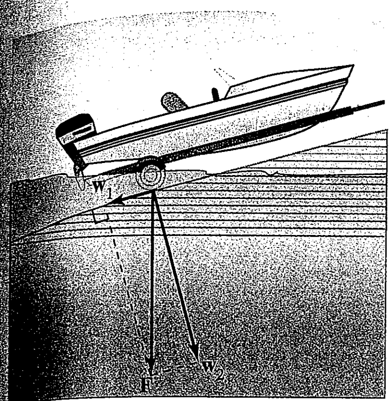
Projections and Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components.** The following physical example enables you to see the usefulness of this procedure.

Consider a boat on an inclined ramp, as shown in Figure 11.28. The force $\mathbf{F}$ due to gravity pulls the boat *down* the ramp and *against* the ramp. These two forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are orthogonal—they are called the vector components of $\mathbf{F}$.

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2 \qquad \text{Vector components of } \mathbf{F}$$

The forces $\mathbf{w}_1$ and $\mathbf{w}_2$ help you analyze the effect of gravity on the boat. For example, $\mathbf{w}_1$ indicates the force necessary to keep the boat from rolling down the ramp, whereas $\mathbf{w}_2$ indicates the force that the tires must withstand.

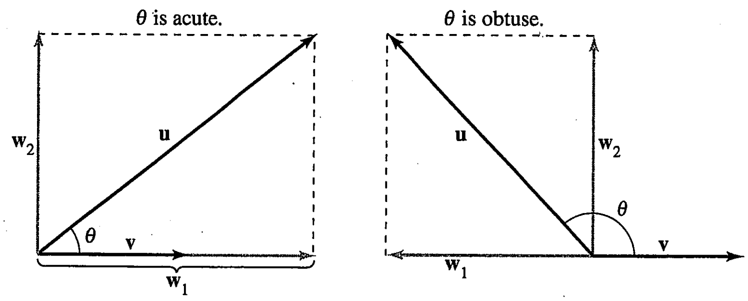


The force due to gravity pulls the boat against the ramp and down the ramp.
Figure 11.28

Definition of Projection and Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v,** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} =$ projection of $\mathbf{u}$ onto $\mathbf{v}$ = vector component of $\mathbf{u}$ along $\mathbf{v}$
$\mathbf{w}_2 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
Figure 11.29

EXAMPLE 4 Finding a Vector Component of u Orthogonal to v

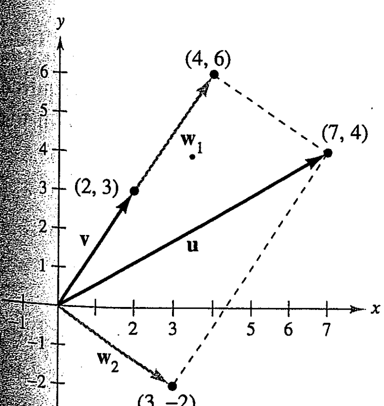
Find the vector component of $\mathbf{u} = \langle 7, 4 \rangle$ that is orthogonal to $\mathbf{v} = \langle 2, 3 \rangle$, given that $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 4, 6 \rangle$ and

$$\mathbf{u} = \langle 7, 4 \rangle = \mathbf{w}_1 + \mathbf{w}_2.$$

Solution Because $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, it follows that $\mathbf{w}_2$ is the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$. So, you have

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1 \\ &= \langle 7, 4 \rangle - \langle 4, 6 \rangle \\ &= \langle 3, -2 \rangle.\end{aligned}$$

Check to see that $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.30.



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.30

From Example 4, you can see that it is easy to find the vector component $\mathbf{w}_2$ once you have found the projection, $\mathbf{w}_1$, of $\mathbf{u}$ onto $\mathbf{v}$. To find this projection, use the dot product given in the theorem below, which you will prove in Exercise 90.

NOTE Note the distinction between the terms "component" and "vector component." For example, using the standard unit vectors with $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, u_1 is the *component* of $\mathbf{u}$ in the direction of $\mathbf{i}$ and $u_1\mathbf{i}$ is the *vector component* in the direction of $\mathbf{i}$.

THEOREM 11.6 Projection Using the Dot Product

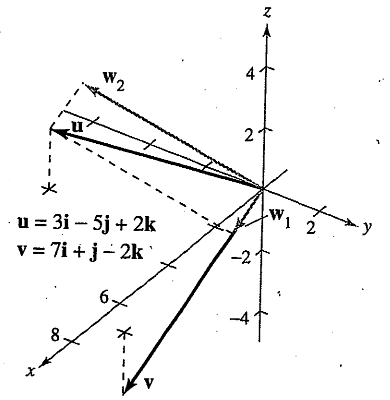
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

The projection of $\mathbf{u}$ onto $\mathbf{v}$ can be written as a scalar multiple of a unit vector in the direction of $\mathbf{v}$. That is,

$$\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} = (k)\frac{\mathbf{v}}{\|\mathbf{v}\|} \implies k = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos \theta.$$

The scalar k is called the **component of u in the direction of v.**



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.31

EXAMPLE 5 Decomposing a Vector into Vector Components

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ for the vectors $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ shown in Figure 11.31.

Solution The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{12}{54}\right)(7\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}.$$

The vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is the vector

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) - \left(\frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}\right) = \frac{13}{9}\mathbf{i} - \frac{47}{9}\mathbf{j} + \frac{22}{9}\mathbf{k}.$$

EXAMPLE 6 Finding a Force

A 600-pound boat sits on a ramp inclined at 30°, as shown in Figure 11.32. What force is required to keep the boat from rolling down the ramp?

Solution Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector $\mathbf{F} = -600\mathbf{j}$. To find the force required to keep the boat from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = \cos 30°\mathbf{i} + \sin 30°\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \qquad \text{Unit vector along ramp}$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is given by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v} = (-600)\left(\frac{1}{2}\right)\mathbf{v} = -300\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).$$

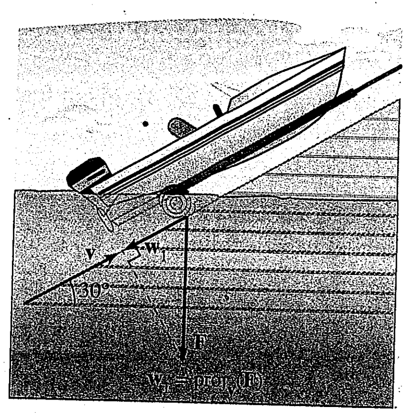
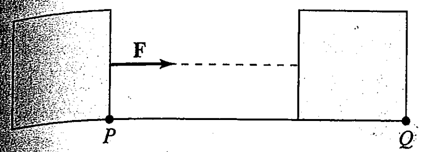


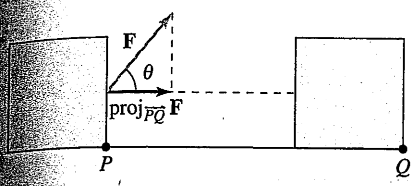
Figure 11.32

The magnitude of this force is 300, and therefore a force of 300 pounds is required to keep the boat from rolling down the ramp.



Work = $\|\mathbf{F}\|\|\overrightarrow{PQ}\|$

(a) Force acts along the line of motion.



Work = $\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\|\overrightarrow{PQ}\|$

(b) Force acts at angle θ with the line of motion.

Figure 11.33

Work

The work W done by the constant force **F** acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|$$

as shown in Figure 11.33(a). If the constant force **F** is not directed along the line of motion, you can see from Figure 11.33(b) that the work W done by the force is

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| = (\cos\theta)\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \mathbf{F}\cdot\overrightarrow{PQ}.$$

This notion of work is summarized in the following definition.

Definition of Work

The work W done by a constant force **F** as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ — Projection form
2. $W = \mathbf{F}\cdot\overrightarrow{PQ}$ — Dot product form

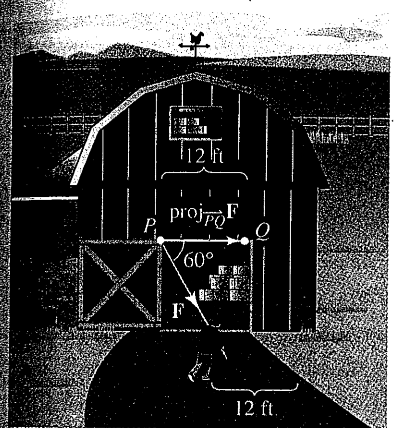


Figure 11.34

EXAMPLE 7 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 11.34. Find the work done in moving the door 12 feet to its closed position.

Solution Using a projection, you can calculate the work as follows.

$$\begin{aligned} W &= \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| && \text{Projection form for work} \\ &= \cos(60^\circ)\,\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \\ &= \frac{1}{2}(50)(12) \\ &= 300 \text{ foot-pounds} \end{aligned}$$

Exercises for Section 11.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find (a) u · v, (b) u · u, (c) $\|\mathbf{u}\|^2$, (d) (u · v)v, and (e) u · (2v).

1. $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle 2, -3\rangle$
2. $\mathbf{u} = \langle 4, 10\rangle$, $\mathbf{v} = \langle -2, 3\rangle$
3. $\mathbf{u} = \langle 5, -1\rangle$, $\mathbf{v} = \langle -3, 2\rangle$
4. $\mathbf{u} = \langle -4, 8\rangle$, $\mathbf{v} = \langle 6, 3\rangle$
5. $\mathbf{u} = \langle 2, -3, 4\rangle$, $\mathbf{v} = \langle 0, 6, 5\rangle$
6. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$
7. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
8. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

In Exercises 9 and 10, find u · v.

9. $\|\mathbf{u}\| = 8$, $\|\mathbf{v}\| = 5$, and the angle between **u** and **v** is $\pi/3$.
10. $\|\mathbf{u}\| = 40$, $\|\mathbf{v}\| = 25$, and the angle between **u** and **v** is $5\pi/6$.

In Exercises 11–18, find the angle θ between the vectors.

11. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, -2\rangle$
12. $\mathbf{u} = \langle 3, 1\rangle$, $\mathbf{v} = \langle 2, -1\rangle$
13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$
14. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$
15. $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 2, 1, -1\rangle$
16. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$
18. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

In Exercises 19–26, determine whether u and v are orthogonal, parallel, or neither.

19. $\mathbf{u} = \langle 4, 0\rangle$, $\mathbf{v} = \langle 1, 1\rangle$
20. $\mathbf{u} = \langle 2, 18\rangle$, $\mathbf{v} = \left\langle \frac{3}{2}, -\frac{1}{6}\right\rangle$

21. $\mathbf{u} = \langle 4, 3 \rangle$
$\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

22. $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$
$\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

23. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

24. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

25. $\mathbf{u} = \langle 2, -3, 1 \rangle$
$\mathbf{v} = \langle -1, -1, -1 \rangle$

26. $\mathbf{u} = \langle \cos\theta, \sin\theta, -1 \rangle$
$\mathbf{v} = \langle \sin\theta, -\cos\theta, 0 \rangle$

In Exercises 27–30, the vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.

27. $(1, 2\ 0), (0, 0, 0), (-2, 1, 0)$

28. $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$

29. $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$

30. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

In Exercises 31–34, find the direction cosines of u and demonstrate that the sum of the squares of the direction cosines is 1.

31. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

32. $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

33. $\mathbf{u} = \langle 0, 6, -4 \rangle$

34. $\mathbf{u} = \langle a, b, c \rangle$

In Exercises 35–38, find the direction angles of the vector.

35. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

36. $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

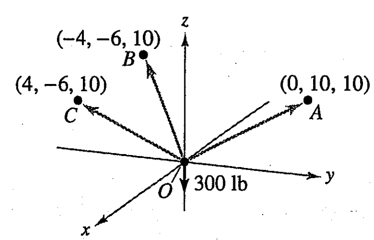
37. $\mathbf{u} = \langle -1, 5, 2 \rangle$

38. $\mathbf{u} = \langle -2, 6, 1 \rangle$

In Exercises 39 and 40, use a graphing utility to find the magnitude and direction angles of the resultant of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with initial points at the origin. The magnitude and terminal point of each vector are given.

	Vector	*Magnitude*	*Terminal Point*
39.	$\mathbf{F}_1$	50 lb	$(10, 5, 3)$
	$\mathbf{F}_2$	80 lb	$(12, 7, -5)$
40.	$\mathbf{F}_1$	300 N	$(-20, -10, 5)$
	$\mathbf{F}_2$	100 N	$(5, 15, 0)$

41. ***Load-Supporting Cables*** A load is supported by three cables, as shown in the figure. Find the direction angles of the load-supporting cable OA.



42. ***Load-Supporting Cables*** The tension in the cable OA in Exercise 41 is 200 newtons. Determine the weight of the load.

In Exercises 43–46, find the component of u that is orthogonal to v, given $\mathbf{w}_1 = \text{proj}_\mathbf{v}\mathbf{u}$.

43. $\mathbf{u} = \langle 6, 7 \rangle$, $\mathbf{v} = \langle 1, 4 \rangle$, $\text{proj}_\mathbf{v}\mathbf{u} = \langle 2, 8 \rangle$

44. $\mathbf{u} = \langle 9, 7 \rangle$, $\mathbf{v} = \langle 1, 3 \rangle$, $\text{proj}_\mathbf{v}\mathbf{u} = \langle 3, 9 \rangle$

45. $\mathbf{u} = \langle 0, 3, 3 \rangle$, $\mathbf{v} = \langle -1, 1, 1 \rangle$, $\text{proj}_\mathbf{v}\mathbf{u} = \langle -2, 2, 2 \rangle$

46. $\mathbf{u} = \langle 8, 2, 0 \rangle$, $\mathbf{v} = \langle 2, 1, -1 \rangle$, $\text{proj}_\mathbf{v}\mathbf{u} = \langle 6, 3, -3 \rangle$

In Exercises 47–50, (a) find the projection of u onto v, and (b) find the vector component of u orthogonal to v.

47. $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle 5, 1 \rangle$

48. $\mathbf{u} = \langle 2, -3 \rangle$, $\mathbf{v} = \langle 3, 2 \rangle$

49. $\mathbf{u} = \langle 2, 1, 2 \rangle$
$\mathbf{v} = \langle 0, 3, 4 \rangle$

50. $\mathbf{u} = \langle 1, 0, 4 \rangle$
$\mathbf{v} = \langle 3, 0, 2 \rangle$

Writing About Concepts

51. Define the dot product of vectors **u** and **v**.

52. State the definition of orthogonal vectors. If vectors are neither parallel nor orthogonal, how do you find the angle between them? Explain.

53. What is known about θ, the angle between two nonzero vectors **u** and **v**, if

(a) $\mathbf{u} \cdot \mathbf{v} = 0$? (b) $\mathbf{u} \cdot \mathbf{v} > 0$? (c) $\mathbf{u} \cdot \mathbf{v} < 0$?

54. Determine which of the following are defined for nonzero vectors **u**, **v**, and **w**. Explain your reasoning.

(a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ (b) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

(c) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$ (d) $\|\mathbf{u}\| \cdot (\mathbf{v} + \mathbf{w})$

55. Describe direction cosines and direction angles of a vector **v**.

56. Give a geometric description of the projection of **u** onto **v**.

57. What can be said about the vectors **u** and **v** if (a) the projection of **u** onto **v** equals **u** and (b) the projection of **u** onto **v** equals **0**?

58. If the projection of **u** onto **v** has the same magnitude as the projection of **v** onto **u**, can you conclude that $\|\mathbf{u}\| = \|\mathbf{v}\|$? Explain.

59. ***Revenue*** The vector $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$ gives the numbers of hamburgers, chicken sandwiches, and cheeseburgers, respectively, sold at a fast-food restaurant in one week. The vector $\mathbf{v} = \langle 1.35, 2.65, 1.85 \rangle$ gives the prices (in dollars) per unit for the three food items. Find the dot product $\mathbf{u} \cdot \mathbf{v}$, and explain what information it gives.

60. ***Revenue*** Repeat Exercise 59 after increasing prices by 4%. Identify the vector operation used to increase prices by 4%.

61. ***Programming*** Given vectors **u** and **v** in component form, write a program for a graphing utility in which the output is (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) the angle between **u** and **v**.

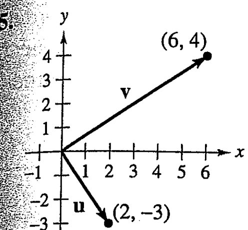
62. ***Programming*** Use the program you wrote in Exercise 61 to find the angle between the vectors $\mathbf{u} = \langle 8, -4, 2 \rangle$ and $\mathbf{v} = \langle 2, 5, 2 \rangle$.

63. ***Programming*** Given vectors **u** and **v** in component form, write a program for a graphing utility in which the output is the component form of the projection of **u** onto **v**.

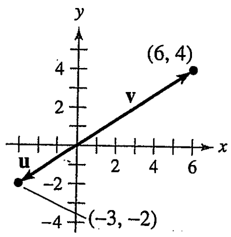
64. *Programming* Use the program you wrote in Exercise 63 to find the projection of **u** onto **v** for **u** = $\langle 5, 6, 2\rangle$ and **v** = $\langle -1, 3, 4\rangle$.

Think About It **In Exercises 65 and 66, use the figure to determine mentally the projection of u onto v. (The coordinates of the terminal points of the vectors in standard position are given.) Verify your results analytically.**

65.



66.



In Exercises 67–70, find two vectors in opposite directions that are orthogonal to the vector u. (The answers are not unique.)

67. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$

68. $\mathbf{u} = -8\mathbf{i} + 3\mathbf{j}$

69. $\mathbf{u} = \langle 3, 1, -2\rangle$

70. $\mathbf{u} = \langle 0, -3, 6\rangle$

71. ***Braking Load*** A 48,000-pound truck is parked on a 10° slope (see figure). Assume the only force to overcome is that due to gravity. Find (a) the force required to keep the truck from rolling down the hill and (b) the force perpendicular to the hill.

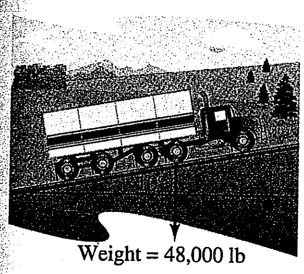


Figure for 71

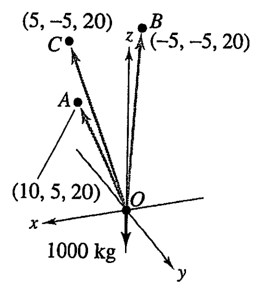


Figure for 72

72. ***Load-Supporting Cables*** Find the magnitude of the projection of the load-supporting cable *OA* onto the positive *z*-axis as shown in the figure.

73. ***Work*** An object is pulled 10 feet across a floor, using a force of 85 pounds. The direction of the force is 60° above the horizontal (see figure). Find the work done.

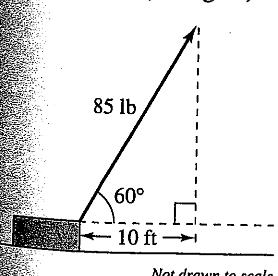


Figure for 73

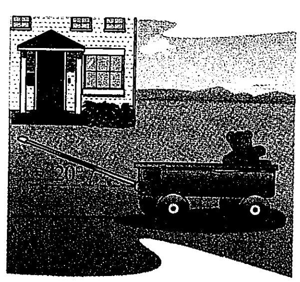


Figure for 74

74. ***Work*** A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal (see figure in left column). Find the work done in pulling the wagon 50 feet.

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

76. If **u** and **v** are orthogonal to **w**, then **u** + **v** is orthogonal to **w**.

77. Find the angle between a cube's diagonal and one of its edges.

78. Find the angle between the diagonal of a cube and the diagonal of one of its sides.

In Exercises 79–82, (a) find the unit tangent vectors to each curve at their points of intersection and (b) find the angles ($0 \le \theta \le 90°$) between the curves at their points of intersection.

79. $y = x^2, \quad y = x^{1/3}$

80. $y = x^3, \quad y = x^{1/3}$

81. $y = 1 - x^2, \quad y = x^2 - 1$

82. $(y + 1)^2 = x, \quad y = x^3 - 1$

83. Use vectors to prove that the diagonals of a rhombus are perpendicular.

84. Use vectors to prove that a parallelogram is a rectangle if and only if its diagonals are equal in length.

85. ***Bond Angle*** Consider a regular tetrahedron with vertices $(0, 0, 0)$, $(k, k, 0)$, $(k, 0, k)$, and $(0, k, k)$, where k is a positive real number.

(a) Sketch the graph of the tetrahedron.

(b) Find the length of each edge.

(c) Find the angle between any two edges.

(d) Find the angle between the line segments from the centroid $(k/2, k/2, k/2)$ to two vertices. This is the bond angle for a molecule such as CH_4 or $PbCl_4$, where the structure of the molecule is a tetrahedron.

86. Consider the vectors

$$\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$$

and

$$\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$$

where $\alpha > \beta$. Find the dot product of the vectors and use the result to prove the identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

87. Prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$.

88. Prove the **Cauchy-Schwarz Inequality** $|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

89. Prove the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

90. Prove Theorem 11.6.