

Section 11.4

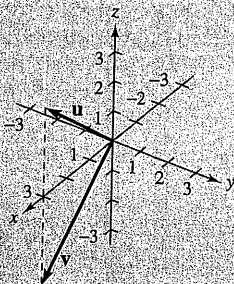
The Cross Product of Two Vectors in Space

- Find the cross product of two vectors in space.
- Use the triple scalar product of three vectors in space.

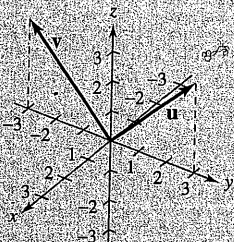
EXPLORATION

Geometric Property of the Cross Product Three pairs of vectors are shown below. Use the definition to find the cross product of each pair. Sketch all three vectors in a three-dimensional system. Describe any relationships among the three vectors. Use your description to write a conjecture about \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$.

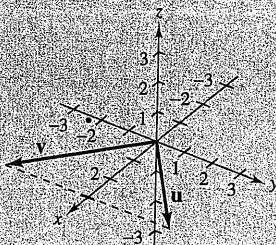
a. $\mathbf{u} = \langle 3, 0, 3 \rangle$, $\mathbf{v} = \langle 3, 0, -3 \rangle$



b. $\mathbf{u} = \langle 0, 3, 3 \rangle$, $\mathbf{v} = \langle 0, -3, 3 \rangle$



c. $\mathbf{u} = \langle 3, 3, 0 \rangle$, $\mathbf{v} = \langle 3, -3, 0 \rangle$



The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section you will study a product that will yield such a vector. It is called the **cross product**, and it is most conveniently defined and calculated using the standard unit vector form. Because the cross product yields a vector, it is also called the **vector product**.

Definition of Cross Product of Two Vectors in Space

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The cross product of \mathbf{u} and \mathbf{v} is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

NOTE Be sure you see that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} && \leftarrow \text{Put "u" in Row 2.} \\ & && \leftarrow \text{Put "v" in Row 3.} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \end{aligned}$$

Note the minus sign in front of the \mathbf{j} -component. Each of the three 2×2 determinants can be evaluated by using the following diagonal pattern.

$$\begin{vmatrix} a & b \\ e & d \end{vmatrix} = ad - bc$$

Here are a couple of examples.

$$\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (4)(3) = -2 - 12 = -14$$

$$\begin{vmatrix} 4 & 0 \\ -6 & 3 \end{vmatrix} = (4)(3) - (0)(-6) = 12$$

NOTATION FOR DOT AND CROSS PRODUCTS

The notation for the dot product and cross product of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called “vector analysis.” The system was a departure from Hamilton’s theory of quaternions.

EXAMPLE 1 Finding the Cross Product

Given $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find each of the following.

a. $\mathbf{u} \times \mathbf{v}$ b. $\mathbf{v} \times \mathbf{u}$ c. $\mathbf{v} \times \mathbf{v}$

Solution

$$\begin{aligned} \mathbf{a.} \quad \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{k} \\ &= (4 - 1)\mathbf{i} - (-2 - 3)\mathbf{j} + (1 + 6)\mathbf{k} \\ &= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{b.} \quad \mathbf{v} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{k} \\ &= (1 - 4)\mathbf{i} - (3 + 2)\mathbf{j} + (-6 - 1)\mathbf{k} \\ &= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k} \end{aligned}$$

Note that this result is the negative of that in part (a).

$$\mathbf{c.} \quad \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. These properties, and several others, are summarized in the following theorem.

THEOREM 11.7 Algebraic Properties of the Cross Product

Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors in space, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Proof To prove Property 1, let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

and

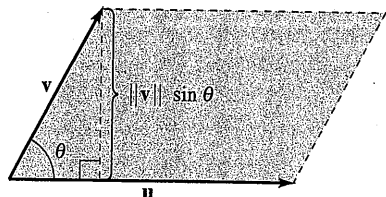
$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$

which implies that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. Proofs of Properties 2, 3, 5, and 6 are left as exercises (see Exercises 57–60).

Note that Property 1 of Theorem 11.7 indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following theorem lists some other *geometric* properties of the cross product of two vectors.

NOTE It follows from Properties 1 and 2 in Theorem 11.8 that if \mathbf{n} is a unit vector orthogonal to both \mathbf{u} and \mathbf{v} , then

$$\mathbf{u} \times \mathbf{v} = \pm(\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta)\mathbf{n}.$$



The vectors \mathbf{u} and \mathbf{v} form adjacent sides of a parallelogram.

Figure 11.35

THEOREM 11.8 Geometric Properties of the Cross Product

Let \mathbf{u} and \mathbf{v} be nonzero vectors in space, and let θ be the angle between \mathbf{u} and \mathbf{v} .

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} .
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if \mathbf{u} and \mathbf{v} are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having \mathbf{u} and \mathbf{v} as adjacent sides.

Proof To prove Property 2, note because $\cos \theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\| \|\mathbf{v}\|)$, it follows that

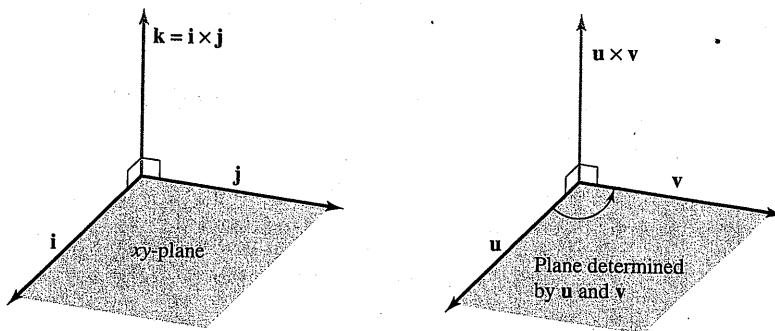
$$\begin{aligned} \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta &= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - \cos^2 \theta} \\ &= \|\mathbf{u}\| \|\mathbf{v}\| \sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2}} \\ &= \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\ &= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} \\ &= \|\mathbf{u} \times \mathbf{v}\|. \end{aligned}$$

To prove Property 4, refer to Figure 11.35, which is a parallelogram having \mathbf{v} and \mathbf{u} as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin \theta$, the area is

$$\begin{aligned} \text{Area} &= (\text{base})(\text{height}) \\ &= \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \\ &= \|\mathbf{u} \times \mathbf{v}\|. \end{aligned}$$

Proofs of Properties 1 and 3 are left as exercises (see Exercises 61 and 62).

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by \mathbf{u} and \mathbf{v} . One way to remember the orientations of the vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors \mathbf{i} , \mathbf{j} , and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 11.36. The three vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*, whereas the three vectors \mathbf{u} , \mathbf{v} , and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system*.



Right-handed systems
Figure 11.36



EXAMPLE 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

Solution The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 11.37, is orthogonal to both \mathbf{u} and \mathbf{v} .

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} && \text{Cross product} \\ &= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k} \end{aligned}$$

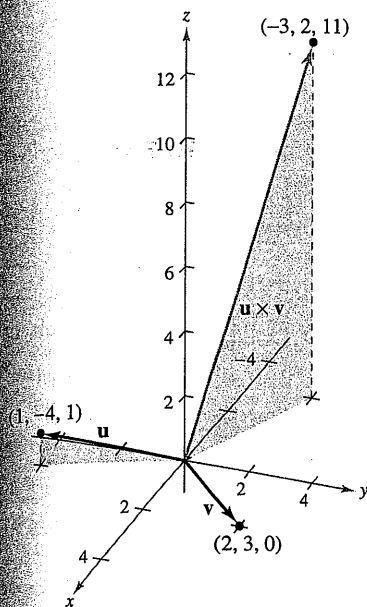
Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

a unit vector orthogonal to both \mathbf{u} and \mathbf{v} is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}.$$

NOTE In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both \mathbf{u} and \mathbf{v} . With that choice, you would have obtained the negative of the unit vector found in the example.



The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} .

Figure 11.37

EXAMPLE 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram, and find its area.

$$\begin{aligned} A &= (5, 2, 0) & B &= (2, 6, 1) \\ C &= (2, 4, 7) & D &= (5, 0, 6) \end{aligned}$$

Solution From Figure 11.38 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\begin{aligned} \overrightarrow{AB} &= -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} & \overrightarrow{CD} &= 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB} \\ \overrightarrow{AD} &= 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} & \overrightarrow{CB} &= 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD} \end{aligned}$$

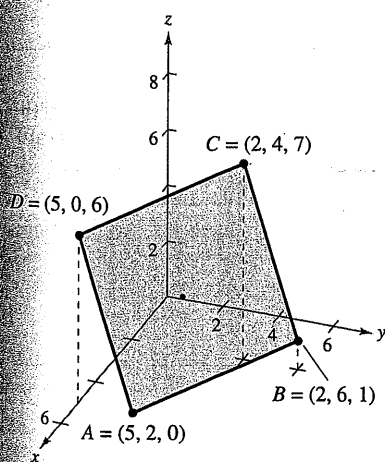
So, \overrightarrow{AB} is parallel to \overrightarrow{CD} and \overrightarrow{AD} is parallel to \overrightarrow{CB} , and you can conclude that the quadrilateral is a parallelogram with \overrightarrow{AB} and \overrightarrow{AD} as adjacent sides. Moreover, because

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} && \text{Cross product} \\ &= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k} \end{aligned}$$

the area of the parallelogram is

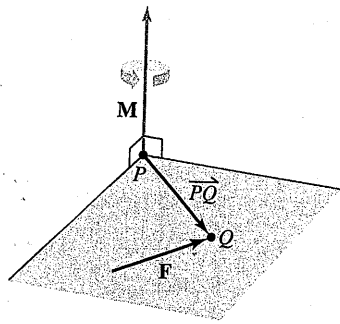
$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can determine whether it is by finding the angle between the vectors \overrightarrow{AB} and \overrightarrow{AD} .

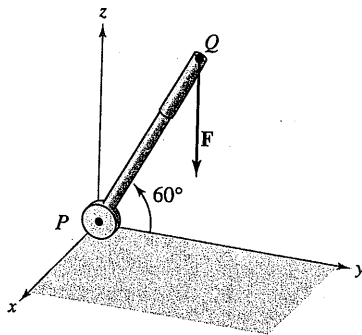


The area of the parallelogram is approximately 32.19.

Figure 11.38



The moment of \mathbf{F} about P
Figure 11.39



A vertical force of 50 pounds is applied at point Q .
Figure 11.40

In physics, the cross product can be used to measure **torque**—the **moment** \mathbf{M} of a force \mathbf{F} about a point P , as shown in Figure 11.39. If the point of application of the force is Q , the moment of \mathbf{F} about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}.$$

Moment of \mathbf{F} about P

The magnitude of the moment \mathbf{M} measures the tendency of the vector \overrightarrow{PQ} to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector \mathbf{M} .

EXAMPLE 4 An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a one-foot lever that is attached to an axle at point P , as shown in Figure 11.40. Find the moment of this force about the point P when $\theta = 60^\circ$.

Solution If you represent the 50-pound force as $\mathbf{F} = -50\mathbf{k}$ and the lever as

$$\overrightarrow{PQ} = \cos(60^\circ)\mathbf{j} + \sin(60^\circ)\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$$

the moment of \mathbf{F} about P is given by

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -50 \end{vmatrix} = -25\mathbf{i}.$$

Moment of \mathbf{F} about P

The magnitude of this moment is 25 foot-pounds.

NOTE In Example 4, note that the moment (the tendency of the lever to rotate about its axle) is dependent on the angle θ . When $\theta = \pi/2$, the moment is $\mathbf{0}$. The moment is greatest when $\theta = 0$.

The Triple Scalar Product

For vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in space, the dot product of \mathbf{u} and $\mathbf{v} \times \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **triple scalar product**, as defined in Theorem 11.9. The proof of this theorem is left as an exercise (see Exercise 56).

THEOREM 11.9 The Triple Scalar Product

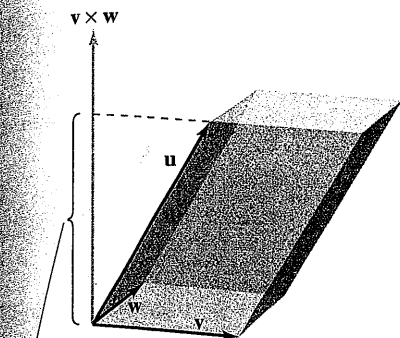
For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

NOTE The value of a determinant is multiplied by -1 if two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. So, the following triple scalar products are equivalent.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

FOR FURTHER INFORMATION To see how the cross product is used to model the torque of the robot arm of a space shuttle, see the article “The Long Arm of Calculus” by Ethan Berkove and Rich Marchand in *The College Mathematics Journal*. To view this article, go to the website www.matharticles.com.



$$\|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\|$$

$$\text{Area of base} = \|\mathbf{v} \times \mathbf{w}\|$$

$$\text{Volume of parallelepiped} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Figure 11.41

If the vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} do not lie in the same plane, the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with \mathbf{u} , \mathbf{v} , and \mathbf{w} as adjacent edges, as shown in Figure 11.41. This is established in the following theorem.

THEOREM 11.10 Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Proof In Figure 11.41, note that

$$\|\mathbf{v} \times \mathbf{w}\| = \text{area of base}$$

and

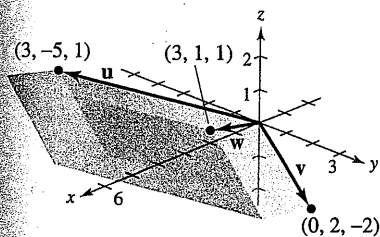
$$\|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\| = \text{height of parallelepiped.}$$

Therefore, the volume is

$$\begin{aligned} V &= (\text{height})(\text{area of base}) = \|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\| \|\mathbf{v} \times \mathbf{w}\| \\ &= \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|} \|\mathbf{v} \times \mathbf{w}\| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

EXAMPLE 5 Volume by the Triple Scalar Product

Find the volume of the parallelepiped shown in Figure 11.42 having $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.



The parallelepiped has a volume of 36.

Figure 11.42

Solution By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| && \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3 \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5) \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1) \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$

A natural consequence of Theorem 11.10 is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is, if the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ have the same initial point, they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Exercises for Section 11.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the cross product of the unit vectors and sketch your result.

- 1. $\mathbf{j} \times \mathbf{i}$
- 2. $\mathbf{i} \times \mathbf{j}$
- 3. $\mathbf{j} \times \mathbf{k}$
- 4. $\mathbf{k} \times \mathbf{j}$
- 5. $\mathbf{i} \times \mathbf{k}$
- 6. $\mathbf{k} \times \mathbf{i}$

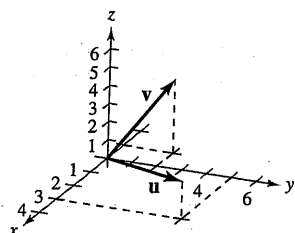
In Exercises 7–10, find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

- 7. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$
 $\mathbf{v} = 3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$
- 8. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{k}$
 $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
- 9. $\mathbf{u} = \langle 7, 3, 2 \rangle$
 $\mathbf{v} = \langle 1, -1, 5 \rangle$
- 10. $\mathbf{u} = \langle 3, -2, -2 \rangle$
 $\mathbf{v} = \langle 1, 5, 1 \rangle$

In Exercises 11–16, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both \mathbf{u} and \mathbf{v} .

- 11. $\mathbf{u} = \langle 2, -3, 1 \rangle$
 $\mathbf{v} = \langle 1, -2, 1 \rangle$
- 12. $\mathbf{u} = \langle -1, 1, 2 \rangle$
 $\mathbf{v} = \langle 0, 1, 0 \rangle$
- 13. $\mathbf{u} = \langle 12, -3, 0 \rangle$
 $\mathbf{v} = \langle -2, 5, 0 \rangle$
- 14. $\mathbf{u} = \langle -10, 0, 6 \rangle$
 $\mathbf{v} = \langle 7, 0, 0 \rangle$
- 15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
 $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
- 16. $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$
 $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

Think About It In Exercises 17–20, use the vectors \mathbf{u} and \mathbf{v} shown in the figure to sketch a vector in the direction of the indicated cross product in a right-handed system.



- 17. $\mathbf{u} \times \mathbf{v}$
- 18. $\mathbf{v} \times \mathbf{u}$
- 19. $(-\mathbf{v}) \times \mathbf{u}$
- 20. $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$

CA In Exercises 21–24, use a computer algebra system to find $\mathbf{u} \times \mathbf{v}$ and a unit vector orthogonal to \mathbf{u} and \mathbf{v} .

- 21. $\mathbf{u} = \langle 4, -3.5, 7 \rangle$
 $\mathbf{v} = \langle -1, 8, 4 \rangle$
- 22. $\mathbf{u} = \langle -8, -6, 4 \rangle$
 $\mathbf{v} = \langle 10, -12, -2 \rangle$
- 23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$
 $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$
- 24. $\mathbf{u} = \frac{2}{3}\mathbf{k}$
 $\mathbf{v} = \frac{1}{2}\mathbf{i} + 6\mathbf{k}$

CA 25. **Programming** Given the vectors \mathbf{u} and \mathbf{v} in component form, write a program for a graphing utility in which the output is $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$.

CA 26. **Programming** Use the program you wrote in Exercise 25 to find $\mathbf{u} \times \mathbf{v}$ and $\|\mathbf{u} \times \mathbf{v}\|$ for $\mathbf{u} = \langle -2, 6, 10 \rangle$ and $\mathbf{v} = \langle 3, 8, 5 \rangle$.

Area In Exercises 27–30, find the area of the parallelogram that has the given vectors as adjacent sides. Use a computer algebra system or a graphing utility to verify your result.

- 27. $\mathbf{u} = \mathbf{j}$
 $\mathbf{v} = \mathbf{j} + \mathbf{k}$
- 28. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
 $\mathbf{v} = \mathbf{j} + \mathbf{k}$
- 29. $\mathbf{u} = \langle 3, 2, -1 \rangle$
 $\mathbf{v} = \langle 1, 2, 3 \rangle$
- 30. $\mathbf{u} = \langle 2, -1, 0 \rangle$
 $\mathbf{v} = \langle -1, 2, 0 \rangle$

Area In Exercises 31 and 32, verify that the points are the vertices of a parallelogram, and find its area.

- 31. $(1, 1, 1), (2, 3, 4), (6, 5, 2), (7, 7, 5)$
- 32. $(2, -3, 1), (6, 5, -1), (3, -6, 4), (7, 2, 2)$

Area In Exercises 33–36, find the area of the triangle with the given vertices. (*Hint:* $\frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$ is the area of the triangle having \mathbf{u} and \mathbf{v} as adjacent sides.)

- 33. $(0, 0, 0), (1, 2, 3), (-3, 0, 0)$
- 34. $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$
- 35. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$
- 36. $(1, 2, 0), (-2, 1, 0), (0, 0, 0)$

37. **Torque** A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a 40° angle with the horizontal (see figure). The crank is 6 inches in length. Find the torque at P .

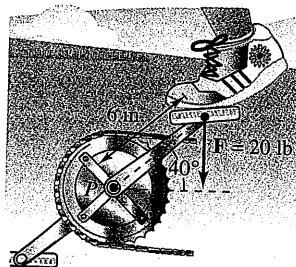


Figure for 37

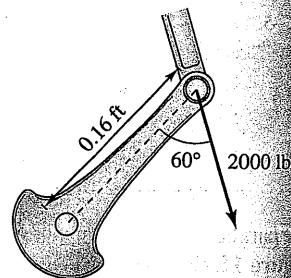


Figure for 38

38. **Torque** Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

CA 39. **Optimization** A force of 60 pounds acts on the pipe wrench shown in the figure on the next page.

- (a) Find the magnitude of the moment about O by evaluating $\|\overrightarrow{OA} \times \mathbf{F}\|$. Use a graphing utility to graph the resulting function of θ .
- (b) Use the result of part (a) to determine the magnitude of the moment when $\theta = 45^\circ$.
- (c) Use the result of part (a) to determine the angle θ when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?

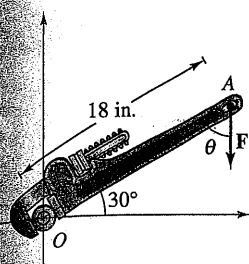


Figure for 39

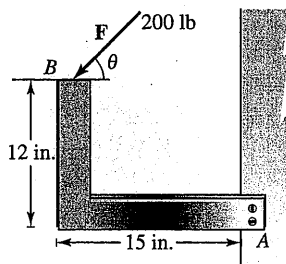


Figure for 40

40. **Optimization** A force of 200 pounds acts on the bracket shown in the figure.

- Determine the vector \overrightarrow{AB} and the vector \mathbf{F} representing the force. (\mathbf{F} will be in terms of θ .)
- Find the magnitude of the moment about A by evaluating $\|\overrightarrow{AB} \times \mathbf{F}\|$.
- Use the result of part (b) to determine the magnitude of the moment when $\theta = 30^\circ$.
- Use the result of part (b) to determine the angle θ when the magnitude of the moment is maximum. At that angle, what is the relationship between the vectors \mathbf{F} and \overrightarrow{AB} ? Is it what you expected? Why or why not?

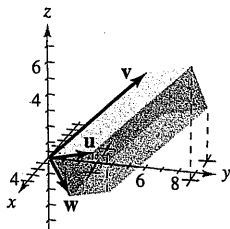
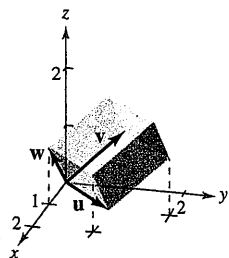
- (e) Use a graphing utility to graph the function for the magnitude of the moment about A for $0^\circ \leq \theta \leq 180^\circ$. Find the zero of the function in the given domain. Interpret the meaning of the zero in the context of the problem.

In Exercises 41–44, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

- | | |
|--|--|
| 41. $\mathbf{u} = \mathbf{i}$ | 42. $\mathbf{u} = \langle 1, 1, 1 \rangle$ |
| $\mathbf{v} = \mathbf{j}$ | $\mathbf{v} = \langle 2, 1, 0 \rangle$ |
| $\mathbf{w} = \mathbf{k}$ | $\mathbf{w} = \langle 0, 0, 1 \rangle$ |
| 43. $\mathbf{u} = \langle 2, 0, 1 \rangle$ | 44. $\mathbf{u} = \langle 2, 0, 0 \rangle$ |
| $\mathbf{v} = \langle 0, 3, 0 \rangle$ | $\mathbf{v} = \langle 1, 1, 1 \rangle$ |
| $\mathbf{w} = \langle 0, 0, 1 \rangle$ | $\mathbf{w} = \langle 0, 2, 2 \rangle$ |

Volume In Exercises 45 and 46, use the triple scalar product to find the volume of the parallelepiped having adjacent edges \mathbf{u} , \mathbf{v} , and \mathbf{w} .

- | | |
|--|--|
| 45. $\mathbf{u} = \mathbf{i} + \mathbf{j}$ | 46. $\mathbf{u} = \langle 1, 3, 1 \rangle$ |
| $\mathbf{v} = \mathbf{j} + \mathbf{k}$ | $\mathbf{v} = \langle 0, 6, 6 \rangle$ |
| $\mathbf{w} = \mathbf{i} + \mathbf{k}$ | $\mathbf{w} = \langle -4, 0, -4 \rangle$ |



Volume In Exercises 47 and 48, find the volume of the parallelepiped with the given vertices (see figures).

- $(0, 0, 0), (3, 0, 0), (0, 5, 1), (3, 5, 1)$
 $(2, 0, 5), (5, 0, 5), (2, 5, 6), (5, 5, 6)$
- $(0, 0, 0), (1, 1, 0), (1, 0, 2), (0, 1, 1)$
 $(2, 1, 2), (1, 1, 3), (1, 2, 1), (2, 2, 3)$

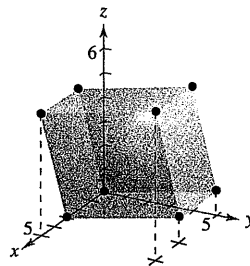


Figure for 47

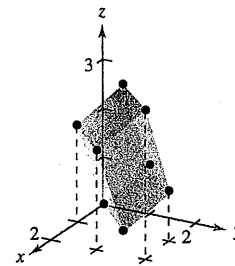


Figure for 48

Writing About Concepts

- Define the cross product of vectors \mathbf{u} and \mathbf{v} .
- State the geometric properties of the cross product.
- If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change? Explain.
- The vertices of a triangle in space are (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) . Explain how to find a vector perpendicular to the triangle.

True or False? In Exercises 53–55, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

- It is possible to find the cross product of two vectors in a two-dimensional coordinate system.
- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
- If $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
- Prove Theorem 11.9.

In Exercises 57–62, prove the property of the cross product.

- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both \mathbf{u} and \mathbf{v} .
- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if \mathbf{u} and \mathbf{v} are scalar multiples of each other.
- Prove $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\|$ if \mathbf{u} and \mathbf{v} are orthogonal.
- Prove $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.