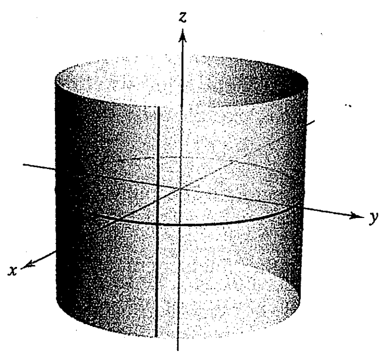
Section 11.6 Surfaces in Space

- **Recognize and write equations for cylindrical surfaces.**
- **Recognize and write equations for quadric surfaces.**
- **Recognize and write equations for surfaces of revolution.**

Cylindrical Surfaces

The first five sections of this chapter contained the vector portion of the preliminary work necessary to study vector calculus and the calculus of space. In this and the next section, you will study surfaces in space and alternative coordinate systems for space. You have already studied two special types of surfaces.

1. Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ Section 11.2
2. Planes: $ax + by + cz + d = 0$ Section 11.5



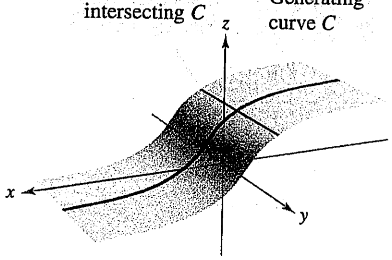
Right circular cylinder: $x^2 + y^2 = a^2$

Rulings are parallel to z-axis.
Figure 11.56

A third type of surface in space is called a **cylindrical surface,** or simply a **cylinder.** To define a cylinder, consider the familiar right circular cylinder shown in Figure 11.56. You can imagine that this cylinder is generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the xy-plane. This circle is called a **generating curve** for the cylinder, as indicated in the following definition.

Definition of a Cylinder

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a **cylinder.** C is called the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are called **rulings.**



Cylinder: Rulings intersect C and are parallel to the given line.
Figure 11.57

NOTE Without loss of generality, you can assume that C lies in one of the three coordinate planes. Moreover, this text restricts the discussion to *right* cylinders—cylinders whose rulings are perpendicular to the coordinate plane containing C, as shown in Figure 11.57.

For the right circular cylinder shown in Figure 11.56, the equation of the generating curve is

$$x^2 + y^2 = a^2.$$ Equation of generating curve in xy-plane

To find an equation for the cylinder, note that you can generate any one of the rulings by fixing the values of x and y and then allowing z to take on all real values. In this sense, the value of z is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2$$ Equation of cylinder in space

Equations of Cylinders

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.

EXAMPLE 1 Sketching a Cylinder

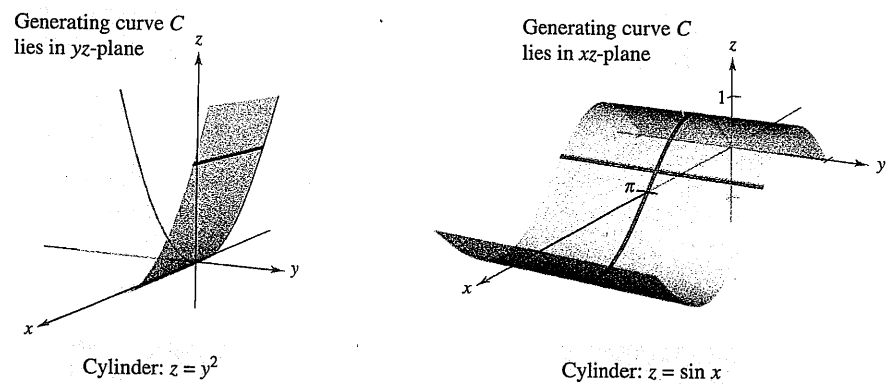
Sketch the surface represented by each equation.

a. $z = y^2$ **b.** $z = \sin x, \quad 0 \le x \le 2\pi$

Solution

a. The graph is a cylinder whose generating curve, $z = y^2$, is a parabola in the yz-plane. The rulings of the cylinder are parallel to the x-axis, as shown in Figure 11.58(a).

b. The graph is a cylinder generated by the sine curve in the xz-plane. The rulings are parallel to the y-axis, as shown in Figure 11.58(b).



(a) Rulings are parallel to x-axis. **(b)** Rulings are parallel to y-axis.
Figure 11.58

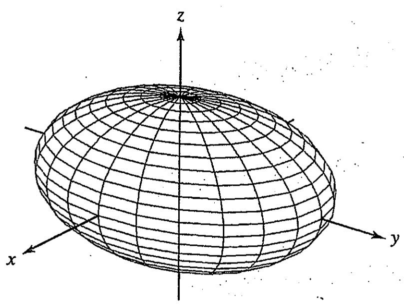
STUDY TIP In the table on pages 812 and 813, only one of several orientations of each quadric surface is shown. If the surface is oriented along a different axis, its standard equation will change accordingly, as illustrated in Examples 2 and 3. The fact that the two types of paraboloids have one variable raised to the first power can be helpful in classifying quadric surfaces. The other four types of basic quadric surfaces have equations that are of *second degree* in all three variables.

Quadric Surfaces

The fourth basic type of surface in space is a **quadric surface.** Quadric surfaces are the three-dimensional analogs of conic sections.

> **Quadric Surface**
>
> The equation of a **quadric surface** in space is a second-degree equation of the form
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$
>
> There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

The intersection of a surface with a plane is called the **trace of the surface** in the plane. To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics. These traces, together with the **standard form** of the equation of each quadric surface, are shown in the table on pages 812 and 813.

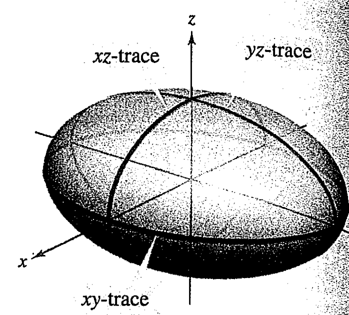


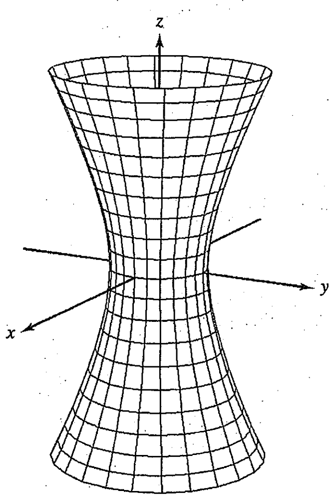
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.



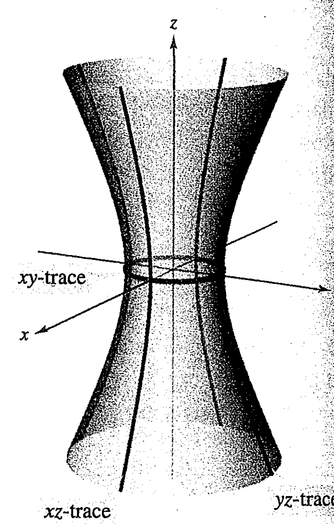


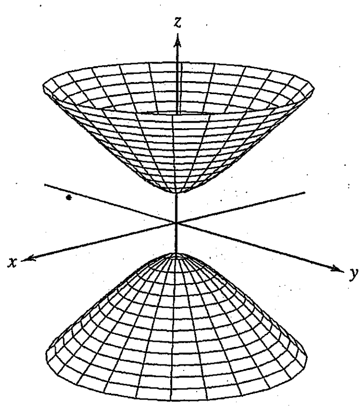
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



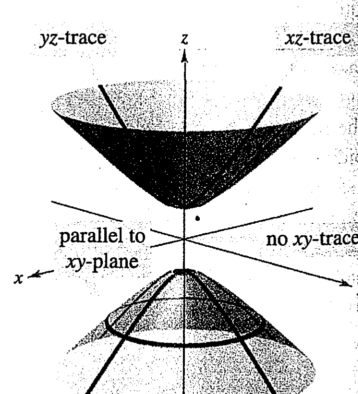


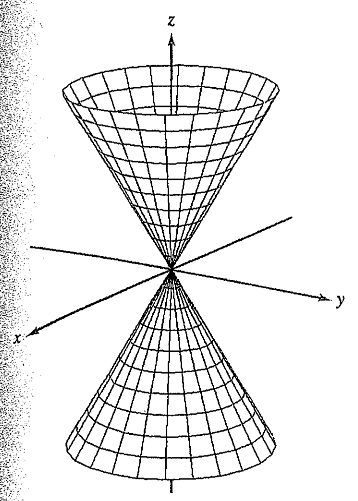
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



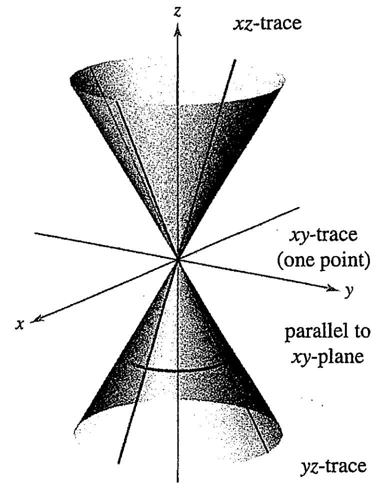


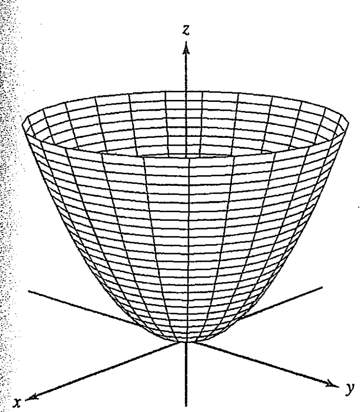
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



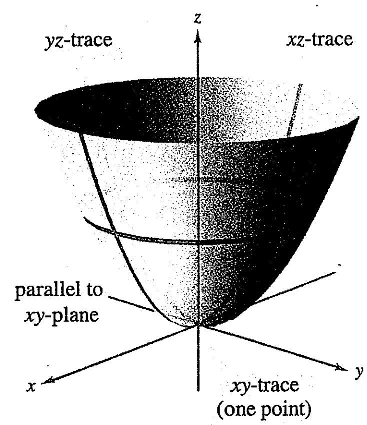


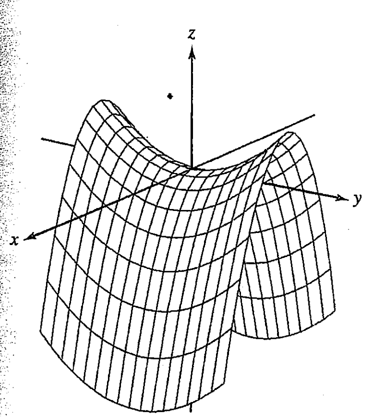
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



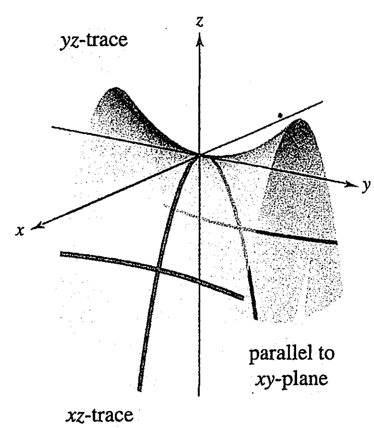


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to xy-plane
Parabola	Parallel to xz-plane
Parabola	Parallel to yz-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



To classify a quadric surface, begin by writing the surface in standard form. Then, determine several traces taken in the coordinate planes *or* taken in planes that are parallel to the coordinate planes.

EXAMPLE 2 Sketching a Quadric Surface

Classify and sketch the surface given by $4x^2 - 3y^2 + 12z^2 + 12 = 0$.

Solution Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0 \qquad \text{Write original equation.}$$

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0 \qquad \text{Divide by } -12.$$

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1 \qquad \text{Standard form}$$

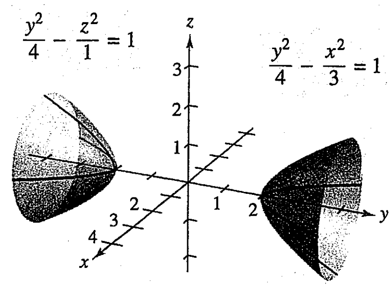
From the table on pages 812 and 813, you can conclude that the surface is a hyperboloid of two sheets with the y-axis as its axis. To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

xy-trace $(z = 0)$: $\dfrac{y^2}{4} - \dfrac{x^2}{3} = 1$ Hyperbola

xz-trace $(y = 0)$: $\dfrac{x^2}{3} + \dfrac{z^2}{1} = -1$ No trace

yz-trace $(x = 0)$: $\dfrac{y^2}{4} - \dfrac{z^2}{1} = 1$ Hyperbola

The graph is shown in Figure 11.59.



Hyperboloid of two sheets: $\frac{y^2}{4} - \frac{x^2}{3} - z^2 = 1$

Figure 11.59

EXAMPLE 3 Sketching a Quadric Surface

Classify and sketch the surface given by $x - y^2 - 4z^2 = 0$.

Solution Because x is raised only to the first power, the surface is a paraboloid. The axis of the paraboloid is the x-axis. In the standard form, the equation is

$$x = y^2 + 4z^2. \qquad \text{Standard form}$$

Some convenient traces are as follows.

xy-trace $(z = 0)$: $x = y^2$ Parabola

xz-trace $(y = 0)$: $x = 4z^2$ Parabola

parallel to yz-plane $(x = 4)$: $\dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$ Ellipse

The surface is an *elliptic* paraboloid, as shown in Figure 11.60.

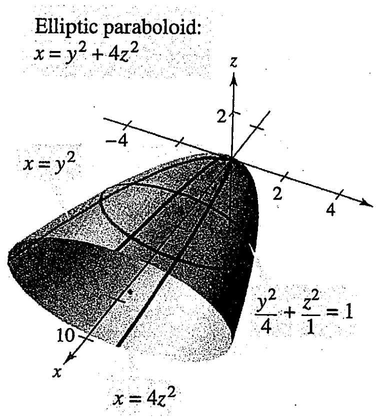


Figure 11.60

Some second-degree equations in x, y, and z do not represent any of the basic types of quadric surfaces. Here are two examples.

$$x^2 + y^2 + z^2 = 0 \qquad \text{Single point}$$

$$x^2 + y^2 = 1 \qquad \text{Right circular cylinder}$$

For a quadric surface not centered at the origin, you can form the standard equation by completing the square, as demonstrated in Example 4.

EXAMPLE 4 A Quadric Surface Not Centered at the Origin

Classify and sketch the surface given by

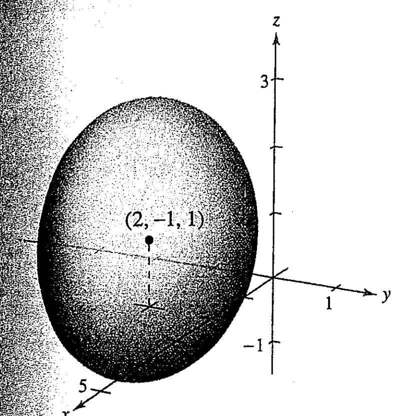
$$x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0.$$

Solution Completing the square for each variable produces the following.

$$(x^2 - 4x + \quad) + 2(y^2 + 2y + \quad) + (z^2 - 2z + \quad) = -3$$
$$(x^2 - 4x + 4) + 2(y^2 + 2y + 1) + (z^2 - 2z + 1) = -3 + 4 + 2 + 1$$
$$(x - 2)^2 + 2(y + 1)^2 + (z - 1)^2 = 4$$
$$\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{2} + \frac{(z - 1)^2}{4} = 1$$

From this equation, you can see that the quadric surface is an ellipsoid that is centered at $(2, -1, 1)$. Its graph is shown in Figure 11.61.

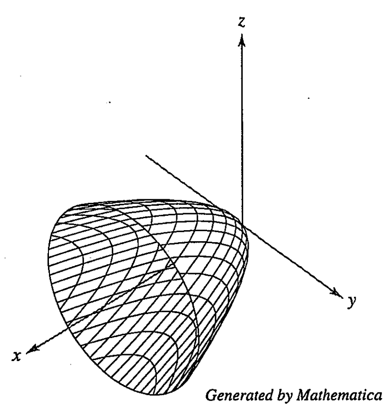
$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{4} = 1$$



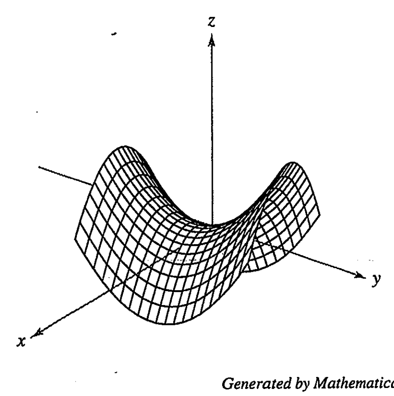
An ellipsoid centered at $(2, -1, 1)$
Figure 11.61

TECHNOLOGY

A computer algebra system can help you visualize a surface in space.* Most of these computer algebra systems create three-dimensional illusions by sketching several traces of the surface and then applying a "hidden-line" routine that blocks out portions of the surface that lie behind other portions of the surface. Two examples of figures that were generated by *Mathematica* are shown below.



Elliptic paraboloid
$x = \frac{y^2}{2} + \frac{z^2}{2}$



Hyperbolic paraboloid
$z = \frac{y^2}{16} - \frac{x^2}{16}$

Using a graphing utility to graph a surface in space requires practice. For one thing, you must know enough about the surface to be able to specify a *viewing window* that gives a representative view of the surface. Also, you can often improve the view of a surface by rotating the axes. For instance, note that the elliptic paraboloid in the figure is seen from a line of sight that is "higher" than the line of sight used to view the hyperbolic paraboloid.

**Some 3-D graphing utilities require surfaces to be entered with parametric equations. For a discussion of this technique, see Section 15.5.*

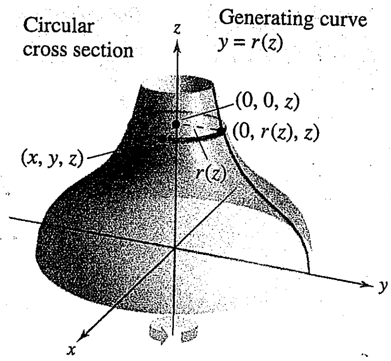


Figure 11.62

Surfaces of Revolution

The fifth special type of surface you will study is called a **surface of revolution.** In Section 7.4, you studied a method for finding the *area* of such a surface. You will now look at a procedure for finding its *equation*. Consider the graph of the **radius function**

$$y = r(z) \qquad \text{Generating curve}$$

in the yz-plane. If this graph is revolved about the z-axis, it forms a surface of revolution, as shown in Figure 11.62. The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$$x^2 + y^2 = [r(z_0)]^2. \qquad \text{Circular trace in plane: } z = z_0$$

Replacing z_0 with z produces an equation that is valid for all values of z. In a similar manner, you can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

Surface of Revolution

If the graph of a radius function r is revolved about one of the coordinate axes, the equation of the resulting surface of revolution has one of the following forms.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

EXAMPLE 5 Finding an Equation for a Surface of Revolution

a. An equation for the surface of revolution formed by revolving the graph of

$$y = \frac{1}{z} \qquad \text{Radius function}$$

about the z-axis is

$$x^2 + y^2 = [r(z)]^2 \qquad \text{Revolved about the } z\text{-axis}$$

$$x^2 + y^2 = \left(\frac{1}{z}\right)^2. \qquad \text{Substitute } 1/z \text{ for } r(z).$$

b. To find an equation for the surface formed by revolving the graph of $9x^2 = y^3$ about the y-axis, solve for x in terms of y to obtain

$$x = \tfrac{1}{3}y^{3/2} = r(y). \qquad \text{Radius function}$$

So, the equation for this surface is

$$x^2 + z^2 = [r(y)]^2 \qquad \text{Revolved about the } y\text{-axis}$$

$$x^2 + z^2 = \left(\tfrac{1}{3}y^{3/2}\right)^2 \qquad \text{Substitute } \tfrac{1}{3}y^{3/2} \text{ for } r(y).$$

$$x^2 + z^2 = \tfrac{1}{9}y^3. \qquad \text{Equation of surface}$$

The graph is shown in Figure 11.63.

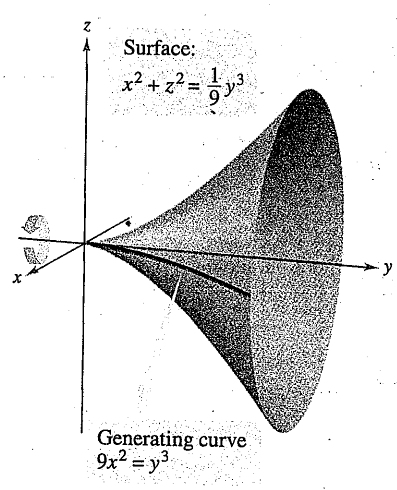


Figure 11.63

The generating curve for a surface of revolution is not unique. For instance, the surface

$$x^2 + z^2 = e^{-2y}$$

can be formed by revolving either the graph of $x = e^{-y}$ about the y-axis or the graph of $z = e^{-y}$ about the y-axis, as shown in Figure 11.64.

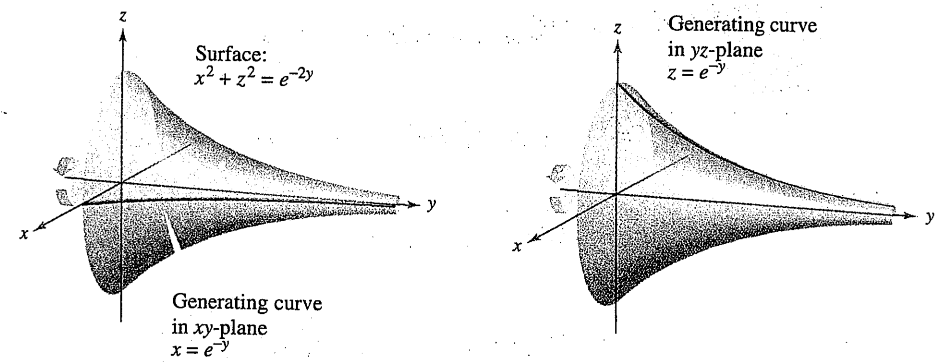


Figure 11.64

EXAMPLE 6 Finding a Generating Curve for a Surface of Revolution

Find a generating curve and the axis of revolution for the surface given by

$$x^2 + 3y^2 + z^2 = 9.$$

Solution You now know that the equation has one of the following forms.

$$x^2 + y^2 = [r(z)]^2 \quad \text{Revolved about } z\text{-axis}$$
$$y^2 + z^2 = [r(x)]^2 \quad \text{Revolved about } x\text{-axis}$$
$$x^2 + z^2 = [r(y)]^2 \quad \text{Revolved about } y\text{-axis}$$

Because the coefficients of x^2 and z^2 are equal, you should choose the third form and write

$$x^2 + z^2 = 9 - 3y^2.$$

The y-axis is the axis of revolution. You can choose a generating curve from either of the following traces.

$$x^2 = 9 - 3y^2 \quad \text{Trace in } xy\text{-plane}$$
$$z^2 = 9 - 3y^2 \quad \text{Trace in } yz\text{-plane}$$

For example, using the first trace, the generating curve is the semiellipse given by

$$x = \sqrt{9 - 3y^2}. \quad \text{Generating curve}$$

The graph of this surface is shown in Figure 11.65.

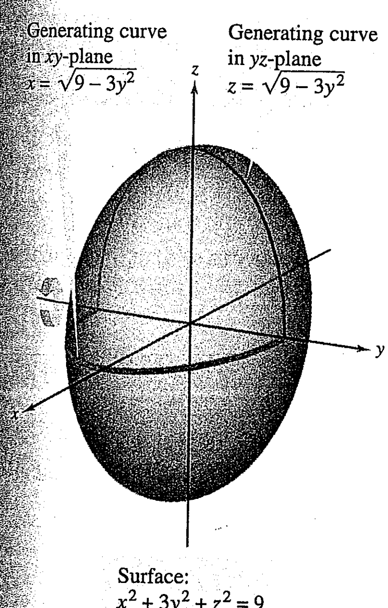
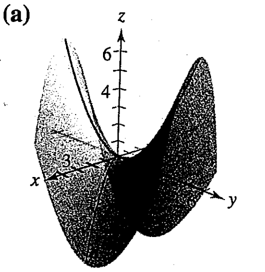


Figure 11.65

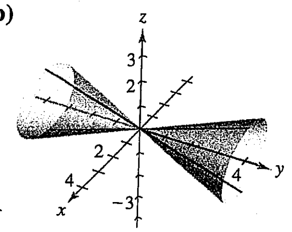
Exercises for Section 11.6

In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

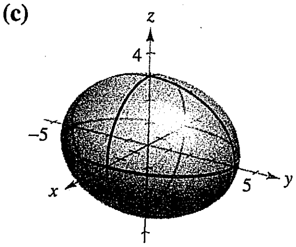
(a)



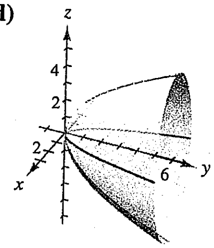
(b)



(c)



(d)



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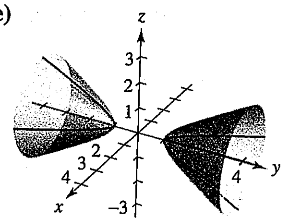
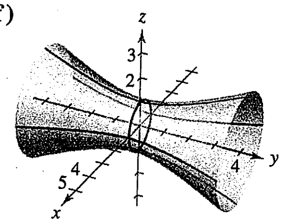


(f)



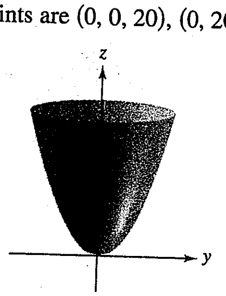
1. $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$
2. $15x^2 - 4y^2 + 15z^2 = -4$
3. $4x^2 - y^2 + 4z^2 = 4$
4. $y^2 = 4x^2 + 9z^2$
5. $4x^2 - 4y + z^2 = 0$
6. $4x^2 - y^2 + 4z = 0$

In Exercises 7–16, describe and sketch the surface.

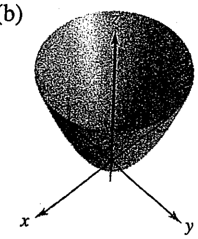
7. $z = 3$
8. $x = 4$
9. $y^2 + z^2 = 9$
10. $x^2 + z^2 = 25$
11. $x^2 - y = 0$
12. $y^2 + z = 4$
13. $4x^2 + y^2 = 4$
14. $y^2 - z^2 = 4$
15. $z - \sin y = 0$
16. $z - e^y = 0$

17. ***Think About It*** The four figures are graphs of the quadric surface $z = x^2 + y^2$. Match each of the four graphs with the point in space from which the paraboloid is viewed. The four points are $(0, 0, 20)$, $(0, 20, 0)$, $(20, 0, 0)$, and $(10, 10, 20)$.

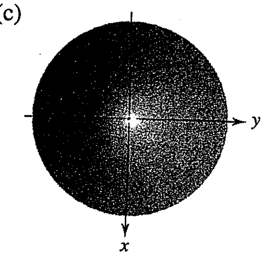
(a)



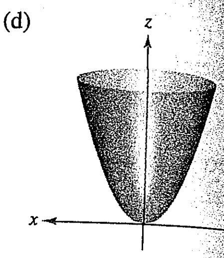
(b)



(c)



(d)



Figures for 17

18. Use a computer algebra system to graph a view of the cylinder $y^2 + z^2 = 4$ from each point.
 (a) $(10, 0, 0)$
 (b) $(0, 10, 0)$
 (c) $(10, 10, 10)$

In Exercises 19–30, identify and sketch the quadric surface. Use a computer algebra system to confirm your sketch.

19. $x^2 + \dfrac{y^2}{4} + z^2 = 1$
20. $\dfrac{x^2}{16} + \dfrac{y^2}{25} + \dfrac{z^2}{25} = 1$
21. $16x^2 - y^2 + 16z^2 = 4$
22. $z^2 - x^2 - \dfrac{y^2}{4} = 1$
23. $x^2 - y + z^2 = 0$
24. $z = x^2 + 4y^2$
25. $x^2 - y^2 + z = 0$
26. $3z = -y^2 + x^2$
27. $z^2 = x^2 + \dfrac{y^2}{4}$
28. $x^2 = 2y^2 + 2z^2$
29. $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$
30. $9x^2 + y^2 - 9z^2 - 54x - 4y - 54z + 4 = 0$

In Exercises 31–40, use a computer algebra system to graph the surface. (*Hint:* It may be necessary to solve for z and acquire two equations to graph the surface.)

31. $z = 2 \sin x$
32. $z = x^2 + 0.5y^2$
33. $z^2 = x^2 + 4y^2$
34. $4y = x^2 + z^2$
35. $x^2 + y^2 = \left(\dfrac{2}{z}\right)^2$
36. $x^2 + y^2 = e^{-z}$
37. $z = 4 - \sqrt{|xy|}$
38. $z = \dfrac{-x}{8 + x^2 + y^2}$
39. $4x^2 - y^2 + 4z^2 = -16$
40. $9x^2 + 4y^2 - 8z^2 = 72$

In Exercises 41–44, sketch the region bounded by the graphs of the equations.

41. $z = 2\sqrt{x^2 + y^2},\ z = 2$
42. $z = \sqrt{4 - x^2},\ y = \sqrt{4 - x^2},\ x = 0,\ y = 0,\ z = 0$
43. $x^2 + y^2 = 1,\ x + z = 2,\ z = 0$
44. $z = \sqrt{4 - x^2 - y^2},\ y = 2z,\ z = 0$

In Exercises 45–50, find an equation for the surface of revolution generated by revolving the curve in the indicated coordinate plane about the given axis.

	Equation of Curve	*Coordinate Plane*	*Axis of Revolution*
45.	$z^2 = 4y$	yz-plane	y-axis
46.	$z = 3y$	yz-plane	y-axis
47.	$z = 2y$	yz-plane	z-axis
48.	$2z = \sqrt{4 - x^2}$	xz-plane	x-axis
49.	$xy = 2$	xy-plane	x-axis
50.	$z = \ln y$	yz-plane	z-axis

In Exercises 51 and 52, find an equation of a generating curve given the equation of its surface of revolution.

51. $x^2 + y^2 - 2z = 0$ **52.** $x^2 + z^2 = \cos^2 y$

Writing About Concepts

53. State the definition of a cylinder.

54. What is meant by the trace of a surface? How do you find a trace?

55. Identify the six quadric surfaces and give the standard form of each.

56. What does the equation $z = x^2$ represent in the xz-plane? What does it represent in three-space?

In Exercises 57 and 58, use the shell method to find the volume of the solid below the surface of revolution and above the xy-plane.

57. The curve $z = 4x - x^2$ in the xz-plane is revolved about the z-axis.

58. The curve $z = \sin y\ (0 \le y \le \pi)$ in the yz-plane is revolved about the z-axis.

In Exercises 59 and 60, analyze the trace when the surface

$$z = \tfrac{1}{2}x^2 + \tfrac{1}{4}y^2$$

is intersected by the indicated planes.

59. Find the lengths of the major and minor axes and the coordinates of the foci of the ellipse generated when the surface is intersected by the planes given by

(a) $z = 2$ and (b) $z = 8$.

60. Find the coordinates of the focus of the parabola formed when the surface is intersected by the planes given by

(a) $y = 4$ and (b) $x = 2$.

In Exercises 61 and 62, find an equation of the surface satisfying the conditions, and identify the surface.

61. The set of all points equidistant from the point $(0, 2, 0)$ and the plane $y = -2$

62. The set of all points equidistant from the point $(0, 0, 4)$ and the xy-plane

63. ***Geography*** Because of the forces caused by its rotation, Earth is an oblate ellipsoid rather than a sphere. The equatorial radius is 3963 miles and the polar radius is 3950 miles. Find an equation of the ellipsoid. (Assume that the center of Earth is at the origin and that the trace formed by the plane $z = 0$ corresponds to the equator.)

64. ***Machine Design*** The top of a rubber bushing designed to absorb vibrations in an automobile is the surface of revolution generated by revolving the curve $z = \frac{1}{2}y^2 + 1\ (0 \le y \le 2)$ in the yz-plane about the z-axis.

(a) Find an equation for the surface of revolution.

(b) All measurements are in centimeters and the bushing is set on the xy-plane. Use the shell method to find its volume.

(c) The bushing has a hole of diameter 1 centimeter through its center and parallel to the axis of revolution. Find the volume of the rubber bushing.

65. Determine the intersection of the hyperbolic paraboloid $z = y^2/b^2 - x^2/a^2$ with the plane $bx + ay - z = 0$. (Assume $a, b > 0$.)

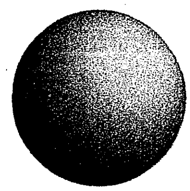
66. Explain why the curve of intersection of the surfaces $x^2 + 3y^2 - 2z^2 + 2y = 4$ and $2x^2 + 6y^2 - 4z^2 - 3x = 2$ lies in a plane.

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

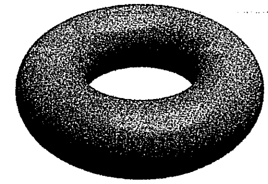
67. A sphere is an ellipsoid.

68. The generating curve for a surface of revolution is unique.

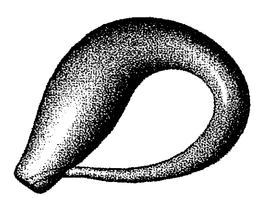
69. ***Think About It*** Three types of classic "topological" surfaces are shown below. The sphere and torus have both an "inside" and an "outside." Does the Klein bottle have both an inside and an outside? Explain.



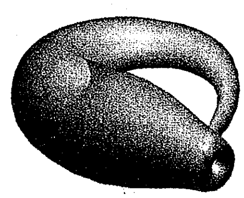
Sphere



Torus



Klein bottle



Klein bottle