Section 12.2 Differentiation and Integration of Vector-Valued Functions

- **Differentiate a vector-valued function.**
- **Integrate a vector-valued function.**

Differentiation of Vector-Valued Functions

In Sections 12.3–12.5, you will study several important applications involving the calculus of vector-valued functions. In preparation for that study, this section is devoted to the mechanics of differentiation and integration of vector-valued functions.

The definition of the derivative of a vector-valued function parallels that given for real-valued functions.

Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(c)$ exists, then $\mathbf{r}$ is **differentiable at** c. If $\mathbf{r}'(c)$ exists for all c in an open interval I, then $\mathbf{r}$ is **differentiable on the interval** I. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

NOTE In addition to $\mathbf{r}'(t)$, other notations for the derivative of a vector-valued function are

$$D_t[\mathbf{r}(t)], \quad \frac{d}{dt}[\mathbf{r}(t)], \quad \text{and} \quad \frac{d\mathbf{r}}{dt}.$$

Differentiation of vector-valued functions can be done on a *component-by-component basis*. To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

Applying the definition of the derivative produces the following.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\{ \left[\frac{f(t + \Delta t) - f(t)}{\Delta t}\right]\mathbf{i} + \left[\frac{g(t + \Delta t) - g(t)}{\Delta t}\right]\mathbf{j} \right\} \\
&= \left\{ \lim_{\Delta t \to 0} \left[\frac{f(t + \Delta t) - f(t)}{\Delta t}\right] \right\}\mathbf{i} + \left\{ \lim_{\Delta t \to 0} \left[\frac{g(t + \Delta t) - g(t)}{\Delta t}\right] \right\}\mathbf{j} \\
&= f'(t)\mathbf{i} + g'(t)\mathbf{j}
\end{aligned}$$

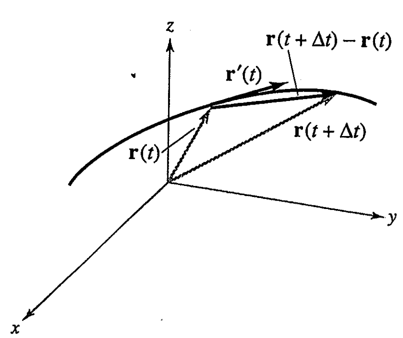


Figure 12.8

This important result is listed in the theorem on the next page. Note that the derivative of the vector-valued function $\mathbf{r}$ is itself a vector-valued function. You can see from Figure 12.8 that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$ and pointing in the direction of increasing t-values.

THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}. \qquad \text{Plane}$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \qquad \text{Space}$$

EXAMPLE 1 Differentiation of Vector-Valued Functions

Find the derivative of each vector-valued function.

a. $\mathbf{r}(t) = t^2\mathbf{i} - 4\mathbf{j}$ **b.** $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + \ln t\mathbf{j} + e^{2t}\mathbf{k}$

Solution Differentiating on a component-by-component basis produces the following.

a. $\mathbf{r}'(t) = 2t\mathbf{i} - 0\mathbf{j}$

$= 2t\mathbf{i}$ Derivative

b. $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{j} + 2e^{2t}\mathbf{k}$ Derivative

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

EXAMPLE 2 Higher-Order Differentiation

For the vector-valued function given by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$, find each of the following.

a. $\mathbf{r}'(t)$ **b.** $\mathbf{r}''(t)$

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ **d.** $\mathbf{r}'(t) \times \mathbf{r}''(t)$

Solution

a. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$ First derivative

b. $\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{k}$

$= -\cos t\mathbf{i} - \sin t\mathbf{j}$ Second derivative

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \sin t \cos t = 0$ Dot product

d. $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix}$ Cross product

$$= \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}\mathbf{k}$$

$$= 2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}$$

Note that the dot product in part (c) is a *real-valued* function, not a vector-valued function.

The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval** I if f', g', and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I.

EXAMPLE 3 Finding Intervals on Which a Curve Is Smooth

Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

is smooth.

Solution The derivative of $\mathbf{r}$ is

$$\mathbf{r}'(t) = (-5\sin t + 5\sin 5t)\mathbf{i} + (5\cos t - 5\cos 5t)\mathbf{j}.$$

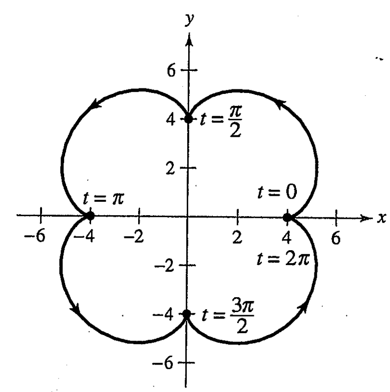
In the interval $[0, 2\pi]$, the only values of t for which

$$\mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j}$$

are $t = 0$, $\pi/2$, π, $3\pi/2$, and 2π. Therefore, you can conclude that C is smooth in the intervals

$$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \text{ and } \left(\frac{3\pi}{2}, 2\pi\right)$$

as shown in Figure 12.9.



$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}$

The epicycloid is not smooth at the points where it intersects the axes.
Figure 12.9

NOTE In Figure 12.9, note that the curve is not smooth at points at which the curve makes abrupt changes in direction. Such points are called **cusps** or **nodes.**

Most of the differentiation rules in Chapter 2 have counterparts for vector-valued functions, and several are listed in the following theorem. Note that the theorem contains three versions of "product rules." Property 3 gives the derivative of the product of a real-valued function f and a vector-valued function $\mathbf{r}$, Property 4 gives the derivative of the dot product of two vector-valued functions, and Property 5 gives the derivative of the cross product of two vector-valued functions (in space). Note that Property 5 applies only to three-dimensional vector-valued functions, because the cross product is not defined for two-dimensional vectors.

THEOREM 12.2 Properties of the Derivative

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Proof To prove Property 4, let

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} \quad \text{and} \quad \mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$$

where f_1, f_2, g_1, and g_2 are differentiable functions of t. Then,

$$\mathbf{r}(t) \cdot \mathbf{u}(t) = f_1(t)f_2(t) + g_1(t)g_2(t)$$

and it follows that

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= f_1(t)f_2'(t) + f_1'(t)f_2(t) + g_1(t)g_2'(t) + g_1'(t)g_2(t) \\ &= [f_1(t)f_2'(t) + g_1(t)g_2'(t)] + [f_1'(t)f_2(t) + g_1'(t)g_2(t)] \\ &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t). \end{aligned}$$

Proofs of the other properties are left as exercises (see Exercises 73–77 and Exercise 80).

EXPLORATION

Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Sketch the graph of $\mathbf{r}(t)$. Explain why the graph is a circle of radius 1 centered at the origin. Calculate $\mathbf{r}(\pi/4)$ and $\mathbf{r}'(\pi/4)$. Position the vector $\mathbf{r}'(\pi/4)$ so that its initial point is at the terminal point of $\mathbf{r}(\pi/4)$. What do you observe? Show that $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant and that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all t. How does this example relate to Property 7 of Theorem 12.2?

EXAMPLE 4 Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find

a. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $D_t[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

Solution

a. Because $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$, you have

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) \\ &\quad + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \frac{1}{t} \\ &= 3 + \frac{1}{t}. \end{aligned}$$

b. Because $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u}''(t) = 2\mathbf{i}$, you have

$$\begin{aligned} D_t[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix}\mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}. \end{aligned}$$

NOTE Try reworking parts (a) and (b) in Example 4 by first forming the dot and cross products and then differentiating to see that you obtain the same results.

Integration of Vector-Valued Functions

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

Definition of Integration of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} \qquad \text{Plane}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k} \qquad \text{Space}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector $\mathbf{C}$. For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t)\,dt$, you obtain three constants of integration

$$\int f(t)\,dt = F(t) + C_1, \qquad \int g(t)\,dt = G(t) + C_2, \qquad \int h(t)\,dt = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$. These three *scalar* constants produce one *vector* constant of integration,

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= [F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k}\\ &= [F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}] + [C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}]\\ &= \mathbf{R}(t) + \mathbf{C}\end{aligned}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

EXAMPLE 5 Integrating a Vector-Valued Function

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

Solution Integrating on a component-by-component basis produces

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt = \frac{t^2}{2}\mathbf{i} + 3t\mathbf{j} + \mathbf{C}.$$

Example 6 shows how to evaluate the definite integral of a vector-valued function.

EXAMPLE 6 Definite Integral of a Vector-Valued Function

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right)dt.$$

Solution

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^{1/3}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{j} + \left(\int_0^1 e^{-t}\,dt\right)\mathbf{k}\\ &= \left[\left(\frac{3}{4}\right)t^{4/3}\right]_0^1 \mathbf{i} + \Big[\ln|t+1|\Big]_0^1 \mathbf{j} + \Big[-e^{-t}\Big]_0^1 \mathbf{k}\\ &= \frac{3}{4}\mathbf{i} + (\ln 2)\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{k}\end{aligned}$$

As with real-valued functions, you can narrow the family of antiderivatives of a vector-valued function $\mathbf{r}'$ down to a single antiderivative by imposing an initial condition on the vector-valued function $\mathbf{r}$. This is demonstrated in the next example.

EXAMPLE 7 The Antiderivative of a Vector-Valued Function

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt\\ &= \left(\int \cos 2t\,dt\right)\mathbf{i} + \left(\int -2\sin t\,dt\right)\mathbf{j} + \left(\int \frac{1}{1+t^2}\,dt\right)\mathbf{k}\\ &= \left(\frac{1}{2}\sin 2t + C_1\right)\mathbf{i} + (2\cos t + C_2)\mathbf{j} + (\arctan t + C_3)\mathbf{k}\end{aligned}$$

Letting $t = 0$ and using the fact that $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, you have

$$\begin{aligned}\mathbf{r}(0) &= (0 + C_1)\mathbf{i} + (2 + C_2)\mathbf{j} + (0 + C_3)\mathbf{k}\\ &= 3\mathbf{i} + (-2)\mathbf{j} + \mathbf{k}.\end{aligned}$$

Equating corresponding components produces

$$C_1 = 3, \qquad 2 + C_2 = -2, \qquad \text{and} \qquad C_3 = 1.$$

So, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin 2t + 3\right)\mathbf{i} + (2\cos t - 4)\mathbf{j} + (\arctan t + 1)\mathbf{k}.$$

Exercises for Section 12.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, sketch the plane curve represented by the vector-valued function, and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of t_0. Position the vectors such that the initial point of $\mathbf{r}(t_0)$ is at the origin and the initial point of $\mathbf{r}'(t_0)$ is at the terminal point of $\mathbf{r}(t_0)$. What is the relationship between $\mathbf{r}'(t_0)$ and the curve?

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, \quad t_0 = 2$

2. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$

3. $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t_0 = 2$

4. $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$

5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$

6. $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}, \quad t_0 = 0$

7. ***Investigation*** Consider the vector-valued function

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$.

(a) Sketch the graph of $\mathbf{r}(t)$. Use a graphing utility to verify your graph.

(b) Sketch the vectors $\mathbf{r}(1/4)$, $\mathbf{r}(1/2)$, and $\mathbf{r}(1/2) - \mathbf{r}(1/4)$ on the graph in part (a).

(c) Compare the vector $\mathbf{r}'(1/4)$ with the vector

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{1/2 - 1/4}.$$

8. ***Investigation*** Consider the vector-valued function

$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$.

(a) Sketch the graph of $\mathbf{r}(t)$. Use a graphing utility to verify your graph.

(b) Sketch the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.25)$, and $\mathbf{r}(1.25) - \mathbf{r}(1)$ on the graph in part (a).

(c) Compare the vector $\mathbf{r}'(1)$ with the vector $\dfrac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1}$.

In Exercises 9 and 10, (a) sketch the space curve represented by the vector-valued function, and (b) sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of t_0.

9. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t_0 = \dfrac{3\pi}{2}$

10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, \quad t_0 = 2$

In Exercises 11–18, find $\mathbf{r}'(t)$.

11. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

13. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

14. $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\sqrt{t}\,\mathbf{j} + \ln t^2\mathbf{k}$

15. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

16. $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2 \rangle$

17. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t \rangle$

18. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0 \rangle$

In Exercises 19–26, find (a) $\mathbf{r}''(t)$ and (b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

19. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

20. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

21. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

22. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

23. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

24. $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

25. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t \rangle$

26. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t \rangle$

In Exercises 27 and 28, a vector-valued function and its graph are given. The graph also shows the unit vectors $\mathbf{r}'(t_0)/\|\mathbf{r}'(t_0)\|$ and $\mathbf{r}''(t_0)/\|\mathbf{r}''(t_0)\|$. Find these two unit vectors and identify them on the graph.

27. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad t_0 = -\frac{1}{4}$

28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, \quad t_0 = \frac{1}{4}$

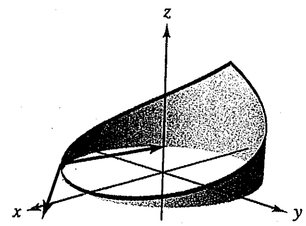


Figure for 27

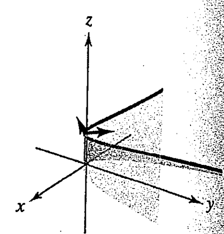


Figure for 28

In Exercises 29–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.

29. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

30. $\mathbf{r}(t) = \dfrac{1}{t - 1}\mathbf{i} + 3t\mathbf{j}$

31. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

32. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

33. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

34. $\mathbf{r}(t) = \dfrac{2t}{8 + t^3}\mathbf{i} + \dfrac{2t^2}{8 + t^3}\mathbf{j}$

35. $\mathbf{r}(t) = (t - 1)\mathbf{i} + \dfrac{1}{t}\mathbf{j} - t^2\mathbf{k}$

36. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

37. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

38. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

In Exercises 39 and 40, use the properties of the derivative to find the following.

(a) $\mathbf{r}'(t)$ (b) $\mathbf{r}''(t)$ (c) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

(d) $D_t[3\mathbf{r}(t) - \mathbf{u}(t)]$ (e) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ (f) $D_t[\|\mathbf{r}(t)\|], \quad t > 0$

39. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

40. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$,

$\mathbf{u}(t) = \dfrac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

In Exercises 41 and 42, find (a) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and (b) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ by differentiating the product, then applying the properties of Theorem 12.2.

41. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$

42. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

In Exercises 43 and 44, find the angle θ between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ as a function of t. Use a graphing utility to graph $\theta(t)$. Use the graph to find any extrema of the function. Find any values of t at which the vectors are orthogonal.

43. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

44. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

In Exercises 45–48, use the definition of the derivative to find $\mathbf{r}'(t)$.

45. $\mathbf{r}(t) = (3t + 2)\mathbf{i} + (1 - t^2)\mathbf{j}$

46. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{3}{t}\mathbf{j} - 2t\mathbf{k}$

47. $\mathbf{r}(t) = \langle t^2, 0, 2t\rangle$

48. $\mathbf{r}(t) = \langle 0, \sin t, 4t\rangle$

In Exercises 49–56, find the indefinite integral.

49. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt$

50. $\displaystyle\int \left(4t^3\,\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\,\mathbf{k}\right) dt$

51. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\,\mathbf{k}\right) dt$

52. $\displaystyle\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right) dt$

53. $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\,\mathbf{k}\right] dt$

54. $\displaystyle\int (e^t\,\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k})\, dt$

55. $\displaystyle\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right) dt$

56. $\displaystyle\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j})\, dt$

In Exercises 57–62, evaluate the definite integral.

57. $\displaystyle\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\, dt$

58. $\displaystyle\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\,\mathbf{k}\right) dt$

59. $\displaystyle\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\, dt$

60. $\displaystyle\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\, dt$

61. $\displaystyle\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\, dt$

62. $\displaystyle\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\, dt$

In Exercises 63–68, find $\mathbf{r}(t)$ for the given conditions.

63. $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

64. $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\,\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

65. $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

66. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

67. $\mathbf{r}'(t) = te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

68. $\mathbf{r}'(t) = \dfrac{1}{1 + t^2}\mathbf{i} + \dfrac{1}{t^2}\mathbf{j} + \dfrac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

Writing About Concepts

69. State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.

70. How do you find the integral of a vector-valued function?

71. The three components of the derivative of the vector-valued function $\mathbf{u}$ are positive at $t = t_0$. Describe the behavior of $\mathbf{u}$ at $t = t_0$.

72. The z-component of the derivative of the vector-valued function $\mathbf{u}$ is 0 for t in the domain of the function. What does this information imply about the graph of $\mathbf{u}$?

In Exercises 73–80, prove the property. In each case, assume $\mathbf{r}$, $\mathbf{u}$, and $\mathbf{v}$ are differentiable vector-valued functions of t, f is a differentiable real-valued function of t, and c is a scalar.

73. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

74. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

75. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

76. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

77. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$

78. $D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

79. $D_t\{\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]\} = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$

80. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is a constant, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

81. ***Particle Motion*** A particle moves in the xy-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.
   (a) Use a graphing utility to graph $\mathbf{r}$. Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

82. ***Particle Motion*** A particle moves in the yz-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2\cos t)\mathbf{j} + (3\sin t)\mathbf{k}$.
   (a) Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

84. The definite integral of a vector-valued function is a real number.

85. $\dfrac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

86. If $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector-valued functions of t, then $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t)$.

87. Consider the vector-valued function

$$\mathbf{r}(t) = (e^t\sin t)\mathbf{i} + (e^t\cos t)\mathbf{j}.$$

Show that $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are always perpendicular to each other.