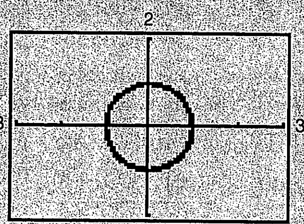
Section 12.3 Velocity and Acceleration

- **Describe the velocity and acceleration associated with a vector-valued function.**
- **Use a vector-valued function to analyze projectile motion.**

> **EXPLORATION**
>
> ***Exploring Velocity*** Consider the circle given by
>
> $\mathbf{r}(t) = (\cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}$.
>
> Use a graphing utility in *parametric* mode to graph this circle for several values of ω. How does ω affect the velocity of the terminal point as it traces out the curve? For a given value of ω, does the speed appear constant? Does the acceleration appear constant? Explain your reasoning.



Velocity and Acceleration

You are now ready to combine your study of parametric equations, curves, vectors, and vector-valued functions to form a model for motion along a curve. You will begin by looking at the motion of an object in the plane. (The motion of an object in space can be developed similarly.)

As an object moves along a curve in the plane, the coordinates x and y of its center of mass are each functions of time t. Rather than using the letters f and g to represent these two functions, it is convenient to write $x = x(t)$ and $y = y(t)$. So, the position vector $\mathbf{r}(t)$ takes the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}. \qquad \text{Position vector}$$

The beauty of this vector model for representing motion is that you can use the first and second derivatives of the vector-valued function $\mathbf{r}$ to find the object's velocity and acceleration. (Recall from the preceding chapter that velocity and acceleration are both vector quantities having magnitude and direction.) To find the velocity and acceleration vectors at a given time t, consider a point $Q(x(t + \Delta t), y(t + \Delta t))$ that is approaching the point $P(x(t), y(t))$ along the curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, as shown in Figure 12.10. As $\Delta t \to 0$, the direction of the vector $\overrightarrow{PQ}$ (denoted by $\Delta\mathbf{r}$) approaches the *direction of motion* at time t.

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

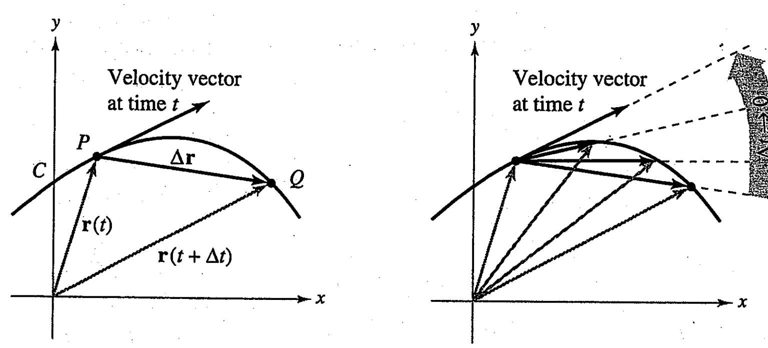
$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

If this limit exists, it is defined to be the **velocity vector** or **tangent vector** to the curve at point P. Note that this is the same limit used to define $\mathbf{r}'(t)$. So, the direction of $\mathbf{r}'(t)$ gives the direction of motion at time t. Moreover, the magnitude of the vector $\mathbf{r}'(t)$

$$\|\mathbf{r}'(t)\| = \|x'(t)\mathbf{i} + y'(t)\mathbf{j}\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

gives the **speed** of the object at time t. Similarly, you can use $\mathbf{r}''(t)$ to find acceleration, as indicated in the definitions at the top of page 849.



As $\Delta t \to 0$, $\dfrac{\Delta\mathbf{r}}{\Delta t}$ approaches the velocity vector.

Figure 12.10

Definitions of Velocity and Acceleration

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$
$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$
$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

For motion along a space curve, the definitions are similar. That is, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, you have

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$
$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$
$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

EXAMPLE 1 Finding Velocity and Acceleration Along a Plane Curve

NOTE In Example 1, note that the velocity and acceleration vectors are orthogonal at any point in time. This is characteristic of motion at a constant speed. (See Exercise 53.)

Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve C described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}. \qquad \text{Position vector}$$

Solution

The velocity vector is

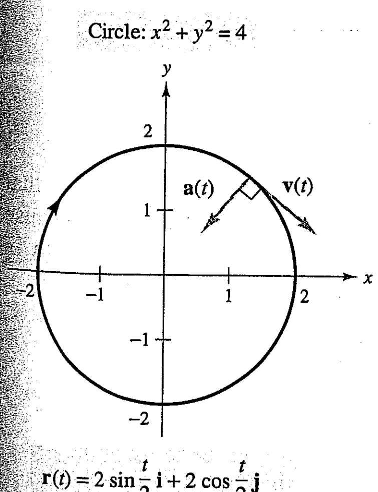
$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}. \qquad \text{Velocity vector}$$

The speed (at any time) is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2}} = 1. \qquad \text{Speed}$$

The acceleration vector is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{1}{2}\sin\frac{t}{2}\mathbf{i} - \frac{1}{2}\cos\frac{t}{2}\mathbf{j}. \qquad \text{Acceleration vector}$$



The particle moves around the circle at a constant speed.
Figure 12.11

The parametric equations for the curve in Example 1 are

$$x = 2\sin\frac{t}{2} \quad \text{and} \quad y = 2\cos\frac{t}{2}.$$

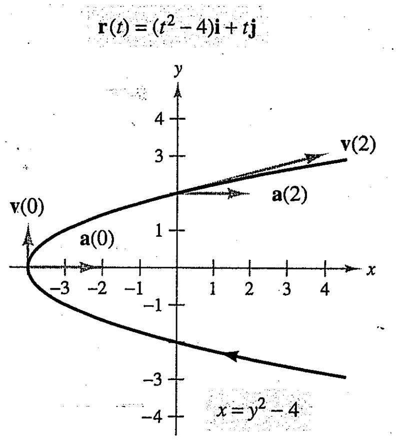
By eliminating the parameter t, you obtain the rectangular equation

$$x^2 + y^2 = 4. \qquad \text{Rectangular equation}$$

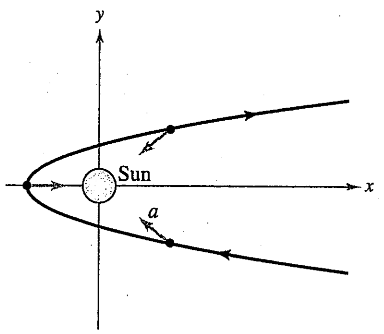
So, the curve is a circle of radius 2 centered at the origin, as shown in Figure 12.11. Because the velocity vector

$$\mathbf{v}(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}$$

has a constant magnitude but a changing direction as t increases, the particle moves around the circle at a constant speed.



At each point on the curve, the acceleration vector points to the right.
Figure 12.12



At each point in the comet's orbit, the acceleration vector points toward the sun.
Figure 12.13

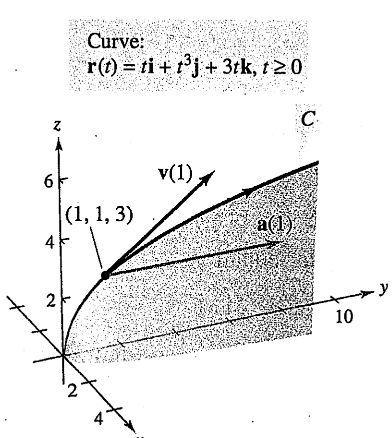


Figure 12.14

EXAMPLE 2 Sketching Velocity and Acceleration Vectors in the Plane

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.

Solution Using the parametric equations $x = t^2 - 4$ and $y = t$, you can determine that the curve is a parabola given by $x = y^2 - 4$, as shown in Figure 12.12. The velocity vector (at any time) is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j} \qquad \text{Velocity vector}$$

and the acceleration vector (at any time) is

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}. \qquad \text{Acceleration vector}$$

When $t = 0$, the velocity and acceleration vectors are given by

$$\mathbf{v}(0) = 2(0)\mathbf{i} + \mathbf{j} = \mathbf{j} \quad \text{and} \quad \mathbf{a}(0) = 2\mathbf{i}.$$

When $t = 2$, the velocity and acceleration vectors are given by

$$\mathbf{v}(2) = 2(2)\mathbf{i} + \mathbf{j} = 4\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{a}(2) = 2\mathbf{i}.$$

For the object moving along the path shown in Figure 12.12, note that the acceleration vector is constant (it has a magnitude of 2 and points to the right). This implies that the speed of the object is decreasing as the object moves toward the vertex of the parabola, and the speed is increasing as the object moves away from the vertex of the parabola.

This type of motion is *not* characteristic of comets that travel on parabolic paths through our solar system. For such comets, the acceleration vector always points to the origin (the sun), which implies that the comet's speed increases as it approaches the vertex of the path and decreases as it moves away from the vertex. (See Figure 12.13.)

EXAMPLE 3 Sketching Velocity and Acceleration Vectors in Space

Sketch the path of an object moving along the space curve C given by

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}, \quad t \geq 0 \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 1$.

Solution Using the parametric equations $x = t$ and $y = t^3$, you can determine that the path of the object lies on the cubic cylinder given by $y = x^3$. Moreover, because $z = 3t$, the object starts at $(0, 0, 0)$ and moves upward as t increases, as shown in Figure 12.14. Because $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}$, you have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} + 3\mathbf{k} \qquad \text{Velocity vector}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{j}. \qquad \text{Acceleration vector}$$

When $t = 1$, the velocity and acceleration vectors are given by

$$\mathbf{v}(1) = \mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{a}(1) = \mathbf{r}''(1) = 6\mathbf{j}.$$

So far in this section, you have concentrated on finding the velocity and acceleration by differentiating the position function. Many practical applications involve the reverse problem—finding the position function for a given velocity or acceleration. This is demonstrated in the next example.

EXAMPLE 4 Finding a Position Function by Integration

An object starts from rest at the point $P(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

Solution From the description of the object's motion, you can deduce the following *initial conditions*. Because the object starts from rest, you have

$$\mathbf{v}(0) = \mathbf{0}.$$

Moreover, because the object starts at the point $(x, y, z) = (1, 2, 0)$, you have

$$\begin{aligned}\mathbf{r}(0) &= x(0)\mathbf{i} + y(0)\mathbf{j} + z(0)\mathbf{k}\\ &= 1\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}\\ &= \mathbf{i} + 2\mathbf{j}.\end{aligned}$$

To find the position function, you should integrate twice, each time using one of the initial conditions to solve for the constant of integration. The velocity vector is

$$\begin{aligned}\mathbf{v}(t) = \int \mathbf{a}(t)\,dt &= \int (\mathbf{j} + 2\mathbf{k})\,dt\\ &= t\mathbf{j} + 2t\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{v}(0) = \mathbf{0}$, you obtain

$$\mathbf{v}(0) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} = \mathbf{0} \implies C_1 = C_2 = C_3 = 0.$$

So, the *velocity* at any time t is

$$\mathbf{v}(t) = t\mathbf{j} + 2t\mathbf{k}. \qquad \text{Velocity vector}$$

Integrating once more produces

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)\,dt &= \int (t\mathbf{j} + 2t\mathbf{k})\,dt\\ &= \frac{t^2}{2}\mathbf{j} + t^2\mathbf{k} + \mathbf{C}\end{aligned}$$

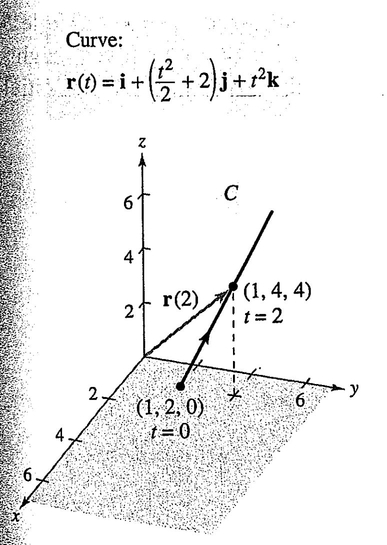
where $\mathbf{C} = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, you have

$$\mathbf{r}(0) = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k} = \mathbf{i} + 2\mathbf{j} \implies C_4 = 1, C_5 = 2, C_6 = 0.$$

So, the *position* vector is

$$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}. \qquad \text{Position vector}$$

The location of the object after $t = 2$ seconds is given by $\mathbf{r}(2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, as shown in Figure 12.15.



The object takes 2 seconds to move from point $(1, 2, 0)$ to point $(1, 4, 4)$ along C.
Figure 12.15

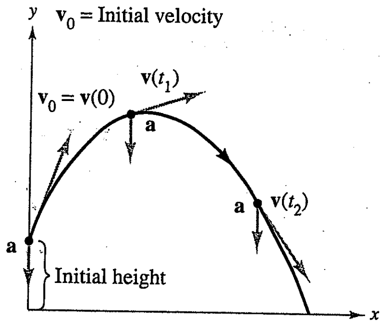


Figure 12.16

Projectile Motion

You now have the machinery to derive the parametric equations for the path of a projectile. Assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the xy-coordinate system with the origin as a point on Earth's surface, as shown in Figure 12.16. For a projectile of mass m, the force due to gravity is

$$\mathbf{F} = -mg\mathbf{j}$$ Force due to gravity

where the gravitational constant is $g = 32$ feet per second per second, or 9.81 meters per second per second. By **Newton's Second Law of Motion,** this same force produces an acceleration $\mathbf{a} = \mathbf{a}(t)$, and satisfies the equation $\mathbf{F} = m\mathbf{a}$. Consequently, the acceleration of the projectile is given by $m\mathbf{a} = -mg\mathbf{j}$, which implies that

$$\mathbf{a} = -g\mathbf{j}.$$ Acceleration of projectile

EXAMPLE 5 Derivation of the Position Function for a Projectile

A projectile of mass m is launched from an initial position $\mathbf{r}_0$ with an initial velocity $\mathbf{v}_0$. Find its position vector as a function of time.

Solution Begin with the acceleration $\mathbf{a}(t) = -g\mathbf{j}$ and integrate twice.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\mathbf{j}\,dt = -gt\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (-gt\mathbf{j} + \mathbf{C}_1)\,dt = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{C}_1t + \mathbf{C}_2$$

You can use the facts that $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ to solve for the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$. Doing this produces $\mathbf{C}_1 = \mathbf{v}_0$ and $\mathbf{C}_2 = \mathbf{r}_0$. Therefore, the position vector is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0.$$ Position vector

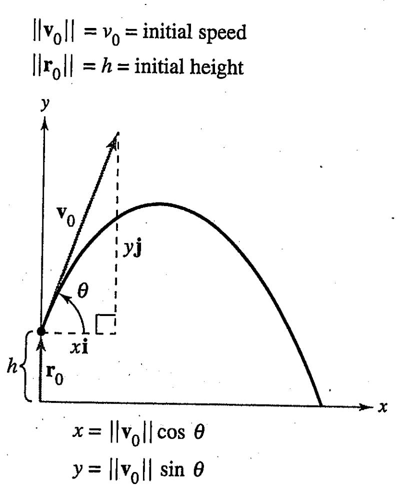


Figure 12.17

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly. Often you are given the initial height h, the initial speed v_0, and the angle θ at which the projectile is launched, as shown in Figure 12.17. From the given height, you can deduce that $\mathbf{r}_0 = h\mathbf{j}$. Because the speed gives the magnitude of the initial velocity, it follows that $v_0 = \|\mathbf{v}_0\|$ and you can write

$$\begin{aligned}\mathbf{v}_0 &= x\mathbf{i} + y\mathbf{j}\\ &= (\|\mathbf{v}_0\|\cos\theta)\mathbf{i} + (\|\mathbf{v}_0\|\sin\theta)\mathbf{j}\\ &= v_0\cos\theta\mathbf{i} + v_0\sin\theta\mathbf{j}.\end{aligned}$$

So, the position vector can be written in the form

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0$$ Position vector

$$= -\frac{1}{2}gt^2\mathbf{j} + tv_0\cos\theta\mathbf{i} + tv_0\sin\theta\mathbf{j} + h\mathbf{j}$$

$$= (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}.$$

THEOREM 12.3 Position Function for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where g is the gravitational constant.

EXAMPLE 6 Describing the Path of a Baseball

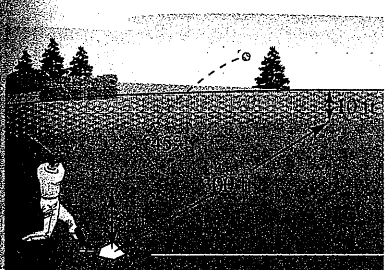


Figure 12.18

A baseball is hit 3 feet above ground level at 100 feet per second and at an angle of 45° with respect to the ground, as shown in Figure 12.18. Find the maximum height reached by the baseball. Will it clear a 10-foot-high fence located 300 feet from home plate?

Solution You are given $h = 3$, $v_0 = 100$, and $\theta = 45°$. So, using $g = 32$ feet per second per second produces

$$\begin{aligned}\mathbf{r}(t) &= \left(100 \cos \frac{\pi}{4}\right)t\mathbf{i} + \left[3 + \left(100 \sin \frac{\pi}{4}\right)t - 16t^2\right]\mathbf{j} \\ &= \left(50\sqrt{2}\,t\right)\mathbf{i} + \left(3 + 50\sqrt{2}\,t - 16t^2\right)\mathbf{j} \\ \mathbf{v}(t) &= \mathbf{r}'(t) = 50\sqrt{2}\,\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}.\end{aligned}$$

The maximum height occurs when

$$y'(t) = 50\sqrt{2} - 32t = 0$$

which implies that

$$\begin{aligned}t &= \frac{25\sqrt{2}}{16} \\ &\approx 2.21 \text{ seconds.}\end{aligned}$$

So, the maximum height reached by the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{25\sqrt{2}}{16}\right)^2 \\ &= \frac{649}{8} \\ &\approx 81 \text{ feet.}\end{aligned}$$

Maximum height when $t \approx 2.21$ seconds

The ball is 300 feet from where it was hit when

$$300 = x(t) = 50\sqrt{2}\,t.$$

Solving this equation for t produces $t = 3\sqrt{2} \approx 4.24$ seconds. At this time, the height of the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(3\sqrt{2}\right) - 16\left(3\sqrt{2}\right)^2 \\ &= 303 - 288 \\ &= 15 \text{ feet.}\end{aligned}$$

Height when $t \approx 4.24$ seconds

Therefore, the ball clears the 10-foot fence for a home run.

Exercises for Section 12.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, the position vector r describes the path of an object moving in the *xy*-plane. Sketch a graph of the path and sketch the velocity and acceleration vectors at the given point.

	Position Function	*Point*
1.	$\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}$	$(3, 0)$
2.	$\mathbf{r}(t) = (6-t)\mathbf{i} + t\mathbf{j}$	$(3, 3)$
3.	$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$	$(4, 2)$
4.	$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$	$(1, 1)$
5.	$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$	$(\sqrt{2}, \sqrt{2})$
6.	$\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$	$(3, 0)$
7.	$\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$	$(\pi, 2)$
8.	$\mathbf{r}(t) = \langle e^{-t}, e^t\rangle$	$(1, 1)$

In Exercises 9–16, the position vector r describes the path of an object moving in space. Find the velocity, speed, and acceleration of the object.

9. $\mathbf{r}(t) = t\mathbf{i} + (2t-5)\mathbf{j} + 3t\mathbf{k}$

10. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

12. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$

13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9-t^2}\,\mathbf{k}$

14. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

15. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$

16. $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$

Linear Approximation **In Exercises 17 and 18, the graph of the vector-valued function r(*t*) and a tangent vector to the graph at $t = t_0$ are given.**

(a) Find a set of parametric equations for the tangent line to the graph at $t = t_0$.

(b) Use the equations for the line to approximate $\mathbf{r}(t_0 + 0.1)$.

17. $\mathbf{r}(t) = \left\langle t, -t^2, \frac{1}{4}t^3\right\rangle, \quad t_0 = 1$

18. $\mathbf{r}(t) = \left\langle t, \sqrt{25-t^2}, \sqrt{25-t^2}\right\rangle, \quad t_0 = 3$

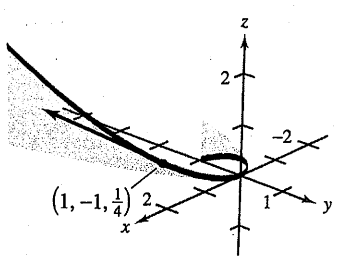


Figure for 17

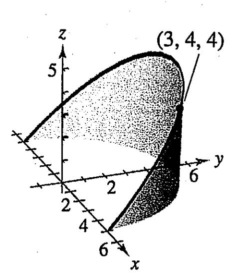


Figure for 18

In Exercises 19–22, use the given acceleration function to find the velocity and position vectors. Then find the position at time $t = 2$.

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}$
$\mathbf{v}(0) = \mathbf{0}, \quad \mathbf{r}(0) = \mathbf{0}$

20. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$
$\mathbf{v}(0) = 4\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}$
$\mathbf{v}(1) = 5\mathbf{j}, \quad \mathbf{r}(1) = \mathbf{0}$

22. $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$
$\mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i}$

Writing About Concepts

23. In your own words, explain the difference between the velocity of an object and its speed.

24. What is known about the speed of an object if the angle between the velocity and acceleration vectors is (a) acute and (b) obtuse?

Projectile Motion **In Exercises 25–40, use the model for projectile motion, assuming there is no air resistance.**

25. Find the vector-valued function for the path of a projectile launched at a height of 10 feet above the ground with an initial velocity of 88 feet per second and at an angle of 30° above the horizontal. Use a graphing utility to graph the path of the projectile.

26. Determine the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of 45° above the horizontal.

27. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball, and how high does it rise?

28. A baseball player at second base throws a ball 90 feet to the player at first base. The ball is thrown 5 feet above the ground with an initial velocity of 50 miles per hour and at an angle of 15° above the horizontal. At what height does the player at first base catch the ball?

29. Eliminate the parameter t from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16\sec^2\theta}{v_0^2}x^2 + (\tan\theta)x + h.$$

30. The path of a ball is given by the rectangular equation

$y = x - 0.005x^2.$

Use the result of Exercise 29 to find the position function. Then find the speed and direction of the ball at the point at which it has traveled 60 feet horizontally.

31. *Modeling Data* After the path of a ball thrown by a baseball player is videotaped, it is analyzed on a television set with a grid covering the screen. The tape is paused three times and the positions of the ball are measured. The coordinates are approximately (0, 6.0), (15, 10.6), and (30, 13.4). (The x-coordinate measures the horizontal distance from the player in feet and the y-coordinate measures the height in feet.)
   (a) Use a graphing utility to find a quadratic model for the data.
   (b) Use a graphing utility to plot the data and graph the model.
   (c) Determine the maximum height of the ball.
   (d) Find the initial velocity of the ball and the angle at which it was thrown.

32. A baseball is hit from a height of 2.5 feet above the ground with an initial velocity of 140 feet per second and at an angle of 22° above the horizontal. Use a graphing utility to graph the path of the ball and determine whether it will clear a 10-foot-high fence located 375 feet from home plate.

33. The SkyDome in Toronto, Ontario has a center field fence that is 10 feet high and 400 feet from home plate. A ball is hit 3 feet above the ground and leaves the bat at a speed of 100 miles per hour.
   (a) The ball leaves the bat at an angle of $\theta = \theta_0$ with the horizontal. Write the vector-valued function for the path of the ball.
   (b) Use a graphing utility to graph the vector-valued function for $\theta_0 = 10°$, $\theta_0 = 15°$, $\theta_0 = 20°$, and $\theta_0 = 25°$. Use the graphs to approximate the minimum angle required for the hit to be a home run.
   (c) Determine analytically the minimum angle required for the hit to be a home run.

34. The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35° with the horizontal.
   (a) Find the speed of the football when it is released.
   (b) Find the maximum height of the football.
   (c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

35. A bale ejector consists of two variable-speed belts at the end of a baler. Its purpose is to toss bales into a trailing wagon. In loading the back of a wagon, a bale must be thrown to a position 8 feet above and 16 feet behind the ejector.
   (a) Find the minimum initial speed of the bale and the corresponding angle at which it must be ejected from the baler.
   (b) The ejector has a fixed angle of 45°. Find the initial speed required.

36. A bomber is flying at an altitude of 30,000 feet at a speed of 540 miles per hour (see figure). When should the bomb be released for it to hit the target? (Give your answer in terms of the angle of depression from the plane to the target.) What is the speed of the bomb at the time of impact?

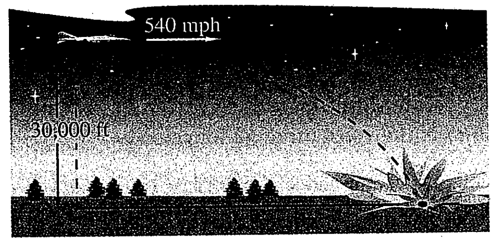


Figure for 36

37. A shot fired from a gun with a muzzle velocity of 1200 feet per second is to hit a target 3000 feet away. Determine the minimum angle of elevation of the gun.

38. A projectile is fired from ground level at an angle of 12° with the horizontal. The projectile is to have a range of 150 feet. Find the minimum initial velocity necessary.

39. Use a graphing utility to graph the paths of a projectile for the given values of θ and v_0. For each case, use the graph to approximate the maximum height and range of the projectile. (Assume that the projectile is launched from ground level.)
   (a) $\theta = 10°$, $v_0 = 66$ ft/sec
   (b) $\theta = 10°$, $v_0 = 146$ ft/sec
   (c) $\theta = 45°$, $v_0 = 66$ ft/sec
   (d) $\theta = 45°$, $v_0 = 146$ ft/sec
   (e) $\theta = 60°$, $v_0 = 66$ ft/sec
   (f) $\theta = 60°$, $v_0 = 146$ ft/sec

40. Find the angle at which an object must be thrown to obtain (a) the maximum range and (b) the maximum height.

Projectile Motion **In Exercises 41 and 42, use the model for projectile motion, assuming there is no resistance. $[a(t) = -9.8$ meters per second per second$]$**

41. Determine the maximum height and range of a projectile fired at a height of 1.5 meters above the ground with an initial velocity of 100 meters per second and at an angle of 30° above the horizontal.

42. A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the minimum velocity necessary.

Cycloidal Motion **In Exercises 43 and 44, consider the motion of a point (or particle) on the circumference of a rolling circle. As the circle rolls, it generates the cycloid**

$$\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$$

where ω is the constant angular velocity of the circle and b is the radius of the circle.

43. Find the velocity and acceleration vectors of the particle. Use the results to determine the times at which the speed of the particle will be (a) zero and (b) maximized.

44. Find the maximum speed of a point on the circumference of an automobile tire of radius 1 foot when the automobile is traveling at 55 miles per hour. Compare this speed with the speed of the automobile.

Circular Motion **In Exercises 45–48, consider a particle moving on a circular path of radius b described by**

$$\mathbf{r}(t) = b\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$$

where $\omega = d\theta/dt$ is the constant angular velocity.

45. Find the velocity vector and show that it is orthogonal to $\mathbf{r}(t)$.

46. (a) Show that the speed of the particle is $b\omega$.
(b) Use a graphing utility in *parametric* mode to graph the circle for $b = 6$. Try different values of ω. Does the graphing utility draw the circle faster for greater values of ω?

47. Find the acceleration vector and show that its direction is always toward the center of the circle.

48. Show that the magnitude of the acceleration vector is $b\omega^2$.

Circular Motion **In Exercises 49 and 50, use the results of Exercises 45–48.**

49. A stone weighing 1 pound is attached to a two-foot string and is whirled horizontally (see figure). The string will break under a force of 10 pounds. Find the maximum speed the stone can attain without breaking the string. (Use $\mathbf{F} = m\mathbf{a}$, where $m = \frac{1}{32}$.)

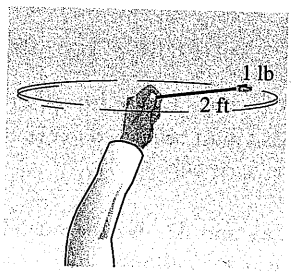


Figure for 49

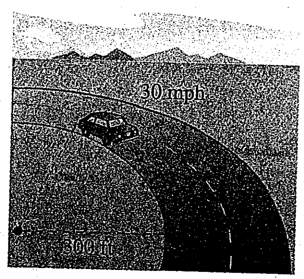


Figure for 50

50. A 3000-pound automobile is negotiating a circular interchange of radius 300 feet at 30 miles per hour (see figure). Assuming the roadway is level, find the force between the tires and the road such that the car stays on the circular path and does not skid. (Use $\mathbf{F} = m\mathbf{a}$, where $m = 3000/32$.) Find the angle at which the roadway should be banked so that no lateral frictional force is exerted on the tires of the automobile.

51. ***Shot-Put Throw*** The path of a shot thrown at an angle θ is

$$\mathbf{r}(t) = (v_0\cos\theta)t\,\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where v_0 is the initial speed, h is the initial height, t is the time in seconds, and g is the acceleration due to gravity. Verify that the shot will remain in the air for a total of

$$t = \frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g}\text{ seconds}$$

and will travel a horizontal distance of

$$\frac{v_0^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right)\text{ feet.}$$

52. ***Shot-Put Throw*** A shot is thrown from a height of $h = 6$ feet with an initial speed of $v_0 = 45$ feet per second and at an angle of $\theta = 42.5°$ with the horizontal. Find the total time of travel and the total horizontal distance traveled.

53. Prove that if an object is traveling at a constant speed, its velocity and acceleration vectors are orthogonal.

54. Prove that an object moving in a straight line at a constant speed has an acceleration of 0.

55. ***Investigation*** An object moves on an elliptical path given by the vector-valued function

$$\mathbf{r}(t) = 6\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}.$$

(a) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.
(b) Use a graphing utility to complete the table.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed					

(c) Graph the elliptical path and the velocity and acceleration vectors at the values of t given in the table in part (b).
(d) Use the results in parts (b) and (c) to describe the geometric relationship between the velocity and acceleration vectors when the speed of the particle is increasing, and when it is decreasing.

56. ***Writing*** Consider a particle moving on the path

$$\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

(a) Discuss any changes in the position, velocity, or acceleration of the particle if its position is given by the vector-valued function $\mathbf{r}_2(t) = \mathbf{r}_1(2t)$.
(b) Generalize the results for the position function $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$.

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

57. The acceleration of an object is the derivative of the speed.

58. The velocity vector points in the direction of motion.

59. When $t = 0$, an object is at the point $(0, 1)$ and has a velocity vector $\mathbf{v}(0) = -\mathbf{i}$. It moves with an acceleration of

$$\mathbf{a}(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}.$$

Show that the path of the object is a circle.