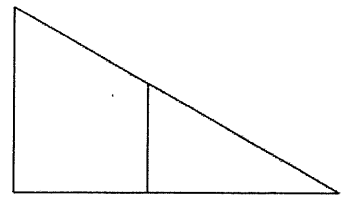
A. More Similar Triangles (Shadow Problem)

$\frac{dy}{dt}$ = R.O.C. of length of shadow;

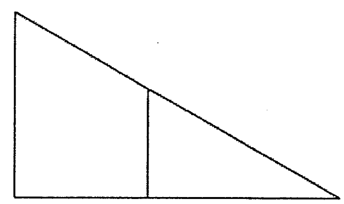
$\frac{dy}{dt} + \frac{dx}{dt}$ = R.O.C. of tip of shadow



Example 1: A man 6 ft tall walks at a rate of 5 ft/s towards the lamp post that is 15 ft above ground. When he is 10 ft from the base of the light

a) At what rate is the length of his shadow changing?

b) At what rate is the tip of his shadow changing?



B. **Linear Approximation** – Approximating nearby y-values on a graph using the tangent line equation

Linear approximation steps:

a) Identify equation and
b) Identify ordered pair of nearest integer x-value
c) Find derivative to determine slope
d) Find equation of tangent line $y - y_1 = m(x - x_1)$
e) Plug decimal value into tangent line equation to approximate function value: $y(x) \approx mx + b$

Example 2: Use linearization to approximate $\sqrt{9.2}$.

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Example 3: Use linearization to approximate $\sqrt[3]{-8.1}$

L'Hopital's rule

L'Hopital's rule says that if $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

Example 4: Find $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Example 5: Find $\lim_{x\to 2}\frac{4x-8}{3x-2}$

Example 6: Find $\lim_{x\to 2}\frac{105x^2-420x+420}{108x^2-432x+432}$

Example 7: Find $\lim_{x\to 5}\frac{x^2-10}{x-5}$

Example 8: Find $\lim_{x\to 0}\frac{\frac{1}{5x+2}-\frac{1}{2}}{x}$

Example 9: Find $\lim_{x\to\infty}\frac{3x+1}{2x^2-5}$

Example 10: Find $\lim_{x\to\infty}\frac{3x^2+1}{2x-5}$

Example 11: Find $\lim_{x\to\infty}\frac{3x+1}{5-2x}$

A. More Similar Triangles (Shadow Problem)

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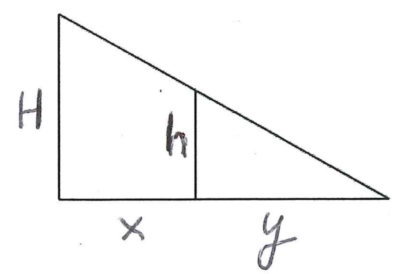


$\frac{h}{H}=\frac{y}{x+y}$

$\frac{dx}{dt}=-5$ ft/s

Example 1: A man 6 ft tall walks at a rate of 5 ft/s towards the lamp post that is 15 ft above ground. When he is 10 ft from the base of the light

$x=10$

a) At what rate is the length of his shadow changing?

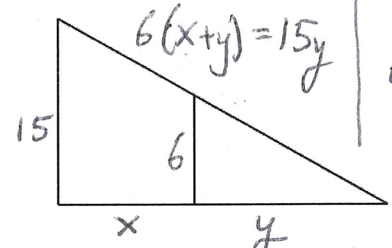
$\frac{dy}{dt}=\frac{-10}{3}$ ft/s

b) At what rate is the tip of his shadow changing?

$\frac{dy}{dx}+\frac{dx}{dt}=-5-\frac{10}{3}=\frac{-25}{3}$ ft/s

a) $\frac{6}{15}=\frac{y}{x+y}$

$6(x+y)=15y$



$6x+6y=15y$

$6x=9y$

$6\frac{dx}{dt}=9\frac{dy}{dt}$

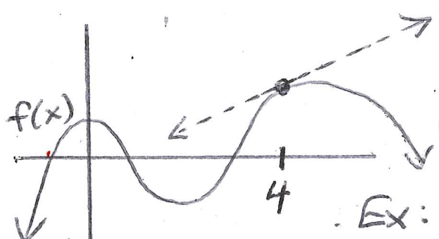
$6(-5)=9\frac{dy}{dt}$

$\frac{-30}{9}=\frac{dy}{dt}$

$\frac{dy}{dt}=\frac{-10}{3}$ ft/s

$\frac{dx}{dt}+\frac{dy}{dt}=-5-\frac{10}{3}=-\frac{25}{3}$ ft/s

B. Linear Approximation – Approximating nearby y-values on a graph using the tangent line equation



Linear approximation steps:

a) Identify equation and
b) Identify ordered pair of nearest integer x-value
c) Find derivative to determine slope
d) Find equation of tangent line $y - y_1 = m(x - x_1)$
e) Plug decimal value into tangent line equation to approximate function value: $y(x) \approx mx + b$

Ex: Use tangent line to approximate f(4.2) instead of plugging x=4.2 into the function.

Example 2: Use linearization to approximate $\sqrt{9.2}$

$y=\sqrt{x}$ $x=9,\ y(9)=\sqrt{9}=3$

point (9, 3)

$y'=\frac{1}{2}x^{-1/2}$

$y'=\frac{1}{2\sqrt{x}}$

$y'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{2(3)}=\frac{1}{6}$

point (9,3)

slope: $m=1/6$

$y-3=\frac{1}{6}(x-9)$

$y=\frac{1}{6}(x-9)+3$

$y(9.2)\approx\frac{1}{6}(9.2-9)+3=$ **3.033**

Linear approximation steps:

a) Identify equation and
b) Identify ordered pair of nearest integer x-value
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Example 3: Use linearization to approximate $\sqrt[3]{-8.1}$

$y = \sqrt[3]{x}$ $x = -8$ $y = \sqrt[3]{-8} = -2$ point: $(-8, -2)$

$y = x^{1/3}$ $y'(-8) = \frac{1}{3(-8)^{2/3}} = \frac{1}{3(-2)^2} = \frac{1}{12}$ slope: $m = 1/12$

$y' = \frac{1}{3}x^{-2/3}$ $y + 2 = \frac{1}{12}(x+8)$ $y(-8.1) \approx \frac{1}{12}(-8.1+8) - 2 = \boxed{-2.008}$

$y' = \frac{1}{3x^{2/3}}$ $y = \frac{1}{12}(x+8) - 2$ $\sqrt[3]{-8.1} \approx -2.008$

L'Hopital's rule

L'Hopital's rule says that if $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ *or* $\pm\frac{\infty}{\infty}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

Example 4: Find $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$ $\lim_{x \to 0} \frac{\frac{1}{2\sqrt{x+1}}}{1} = \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}}$

$g'(x) = 1$

Example 5: Find $\lim_{x \to 2} \frac{4x-8}{3x-2}$ $\frac{0}{4} = \boxed{0}$

Example 6: Find $\lim_{x \to 2} \frac{105x^2 - 420x + 420}{108x^2 - 432x + 432}$ $= \frac{0}{0}$

$f'(x) = 210x - 420$ $\lim_{x \to 2} \frac{210x - 420}{216x - 432} = \frac{0}{0}$

$g'(x) = 216x - 432$

$\lim_{x \to 2} \frac{210}{216} = \frac{210}{216} = \boxed{\frac{35}{36}}$

Example 7: Find $\lim_{x \to 5} \frac{x^2 - 10}{x - 5}$ $\frac{25-10}{5-5} = \frac{15}{0}$ undefined

DNE

Example 8: Find $\lim_{x \to 0} \frac{\frac{1}{5x+2} - \frac{1}{2}}{x}$ $\frac{0}{0}$ $\rightarrow (5x+2)^{-1}$

$f'(x) = -1(5x+2)^{-2}(5)$ $\lim_{x \to 0} \frac{-5}{(5x+2)^2} = \frac{-5}{2^2} = \boxed{\frac{-5}{4}}$

$g'(x) = 1$

Example 9: Find $\lim_{x \to \infty} \frac{3x+1}{2x^2 - 5}$ $\frac{\infty}{\infty} = \frac{3}{4x} \rightarrow \frac{0}{4} = \boxed{0}$

Example 10: Find $\lim_{x \to \infty} \frac{3x^2 + 1}{2x - 5}$ $\frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{6x}{2} \rightarrow \infty$

Example 11: Find $\lim_{x \to \infty} \frac{3x+1}{5-2x}$ $= \frac{\infty}{-\infty}$

$\lim_{x \to \infty} \frac{3}{-2} = \boxed{\frac{-3}{2}}$

AP Calculus AB **Linear Approximation Homework**

1. Find the local linear approximation of $f(x) = \sqrt{x}$ and at $a = 1$ and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.

2. Find the local linear approximation of $f(x) = \dfrac{1}{\sqrt{x}}$ at a = 4 and use it to approximate $\dfrac{1}{\sqrt{3.9}}$ and $\dfrac{1}{\sqrt{4.1}}$.

3. Use an appropriate local linear approximation to estimate the value of the given quantity.

a. $\sqrt{36.1}$

b. $\dfrac{1}{10.1}$

Use an appropriate local linear approximation to estimate the value of the given quantity.

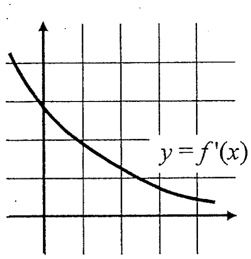
c. $(1.97)^6$

d. $\sqrt[4]{15.8}$

e. $\frac{1}{(3.9)^2}$

f. $\frac{1}{1.9\sqrt{1.9}}$

4. Suppose the only information we have about a function f is that $f(1) = 5$ and the graph of its **derivative** is as shown below.
   a. Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
   b. Are your estimates in part a too large or too small? Explain why.



AP Calculus AB **Linear Approximation Homework**

1. Find the local linear approximation of $f(x) = \sqrt{x}$ and at $a = 1$ and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.

$y=\sqrt{x}$, use $x=1$

$y(1)=\sqrt{1}=1$ $(1,1)$

$y'=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$y'(1)=\frac{1}{2\sqrt{1}}=\frac{1}{2}$

point: $(1,1)$
slope: $m=1/2$

$y-1=\frac{1}{2}(x-1)$

$y=\frac{1}{2}(x-1)+1$

$\sqrt{0.9}\approx\frac{1}{2}(0.9-1)+1=0.95$

$\sqrt{1.1}\approx\frac{1}{2}(1.1-1)+1=1.05$

2. Find the local linear approximation of $f(x)=\frac{1}{\sqrt{x}}$ at a = 4 and use it to approximate $\frac{1}{\sqrt{3.9}}$ and $\frac{1}{\sqrt{4.1}}$.

$y=\frac{1}{\sqrt{x}}$ $y(4)=\frac{1}{\sqrt{4}}=\frac{1}{2}$ $(4,1/2)$

$y=x^{-1/2}$

$y'=\frac{-1}{2}x^{-3/2}$

$y'=\frac{-1}{2x^{3/2}}$

$y'(4)=\frac{-1}{2(4)^{3/2}}=\frac{-1}{2(2)^3}=\frac{-1}{16}$ $m=-1/16$

point: $(4,1/2)$ slope: $m=-1/16$

$y-1/2=-\frac{1}{16}(x-4)$

$y=\frac{-1}{16}(x-4)+1/2$

$\frac{1}{\sqrt{3.9}}\approx\frac{-1}{16}(3.9-4)+1/2=$ **0.506**

$\frac{1}{\sqrt{4.1}}\approx\frac{-1}{16}(4.1-4)+1/2=$ **0.4938**

3. Use an appropriate local linear approximation to estimate the value of the given quantity.

a. $\sqrt{36.1}$ $y=\sqrt{x}$ at $x=36$ $y(36)=\sqrt{36}=6$ point: $(36,6)$

$y=x^{1/2}$

$y'=\frac{1}{2}x^{-1/2}$

$y'=\frac{1}{2\sqrt{x}}$

$y'(36)=\frac{1}{2\sqrt{36}}=\frac{1}{2(6)}$

$y'(36)=\frac{1}{12}$

point: $(36,6)$
slope: $m=1/12$

$y-6=\frac{1}{12}(x-36)$

$y=\frac{1}{12}(x-36)+6$

$\sqrt{36.1}\approx\frac{1}{12}(36.1-36)+6\approx$ **6.008**

b. $\frac{1}{10.1}$ $y(10)=\frac{1}{10}$ $(10,\frac{1}{10})$

$y=\frac{1}{x}=x^{-1}$

$y'(x)=-1x^{-2}$

$y'(x)=\frac{-1}{x^2}$

$y'(10)=\frac{-1}{10^2}=\frac{-1}{100}$

slope: $m=\frac{-1}{100}$

$y-\frac{1}{10}=\frac{-1}{100}(x-10)$

$y=\frac{-1}{100}(x-10)+\frac{1}{10}$

$\frac{1}{10.1}\approx\frac{-1}{100}(10.1-10)+\frac{1}{10}$

$\approx$ **0.099**

Use an appropriate local linear approximation to estimate the value of the given quantity.

c. $(1.97)^6$

$y = x^6$

$y(2) = 2^6 = 64$

point $(2, 64)$

$y' = 6x^5$

$y'(2) = 6(2)^5$

$y'(2) = 192$

$y - 64 = 192(x-2)$

$y = 192(x-2) + 64$

$(1.97)^6 \approx 192(1.97-2) + 64$

$\approx \boxed{58.24}$

d. $\sqrt[4]{15.8}$

$y = \sqrt[4]{x}$

$y(16) = \sqrt[4]{16} = 2$

point: $(16, 2)$

$y(x) = x^{1/4}$

$y' = \frac{1}{4}x^{-3/4}$

$y' = \frac{1}{4x^{3/4}}$

$y'(16) = \frac{1}{4(16)^{3/4}}$

$y'(16) = \frac{1}{32}$

point: $(16, 2)$

slope: $m = \frac{1}{32}$

$y - 2 = \frac{1}{32}(x-16)$

$y = \frac{1}{32}(x-16) + 2$

$\sqrt[4]{15.8} \approx \frac{1}{32}(15.8-16) + 2$

$\approx \boxed{1.99375}$

e. $\frac{1}{(3.9)^2}$

$y = \frac{1}{x^2} = x^{-2}$

$y(4) = \frac{1}{4^2} = \frac{1}{16}$

point: $(4, \frac{1}{16})$

$y' = -2x^{-3}$

$y' = \frac{-2}{x^3}$

$y'(4) = \frac{-2}{4^3}$

$y' = \frac{-2}{64} = \frac{-1}{32}$

point: $(4, 1/16)$

slope: $m = \frac{-1}{32}$

$y - \frac{1}{16} = \frac{-1}{32}(x-4)$

$y = \frac{-1}{32}(x-4) + \frac{1}{16}$

$\frac{1}{(3.9)^2} \approx \frac{-1}{32}(3.9-4) + \frac{1}{16}$

$\approx \boxed{0.066}$

f. $\frac{1}{1.9\sqrt{1.9}}$

$y = \frac{1}{x\sqrt{x}} = \frac{1}{x \cdot x^{1/2}} = x^{-3/2}$

$y' = \frac{-3}{2}x^{-5/2} = \frac{-3}{2x^{5/2}}$

$y(2) = \frac{1}{2\sqrt{2}} =$

$y'(2) = \frac{-3}{2(2)^{5/2}} = \frac{-3}{8\sqrt{2}}$

$y - \frac{1}{2\sqrt{2}} = \frac{-3}{8\sqrt{2}}(x-2)$

$y = \frac{-3}{8\sqrt{2}}(x-2) + \frac{1}{2\sqrt{2}}$

$\frac{1}{1.9\sqrt{1.9}} \approx \frac{-3}{8\sqrt{2}}(1.9-2) + \frac{1}{2\sqrt{2}} \approx \boxed{0.3805}$

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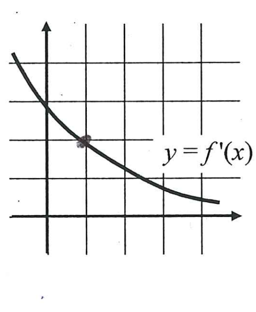

$f'(1) = 2$

point: $(1, 5)$

slope: $m = 2$

$y - 5 = 2(x-1)$

$y = 2(x-1) + 5$

$f(0.9) \approx 2(0.9-1) + 5 \approx \boxed{4.8}$

$f(1.1) \approx 2(1.1-1) + 5 \approx \boxed{5.2}$