Contest # 4 — *Answers & Solutions* — 1/7/20

Problem 4-1

$\dfrac{(x-2019)(x+2019)}{x-2019} = x+2019 = 2020$, so $x = \boxed{1}$.

Problem 4-2

Start working down from 100, writing the primes until two are seaparated by 4. The primes are 97, 89, 83, 79, . . . , and that is all that is needed. Since 83 and 79 differ by 4, the answer is $83 + 79 = \boxed{162}$.

Problem 4-3

Since $2^{22}-2$ is divisible by 2 (the smallest prime), the largest proper divisor of $2^{22}-2$ is $(2^{22}-2)/2 = 2^{21}-1$, so $k = \boxed{21}$.

Problem 4-4

Since the base and altitude of both large unshaded triangles are the same as the base and altitude of the parallelogram, the area of each unshaded large triangle is 22, half the area of the parallelogram. Since the shaded region's total area is 14, the unshaded region's total area is 30. Remove either large unshaded triangle from the total unshaded region to show that the area of each small unshaded triangle is $30-22 = 8$. The unshaded region common to the large unshaded triangles has an area of $30-8-8 = \boxed{14}$.

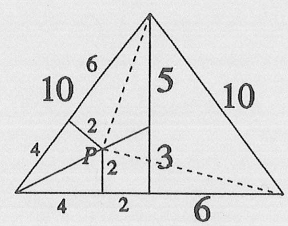
[NOTE: It can be proved that the unshaded common region always has the same area as the shaded region.]

Problem 4-5

This question could be asked about any number of positive integers $n \geq 3$ having the same sum and product. A soluiotn that would work in any case is to make all the numbers 1 except the last two, which would be 2 and n, resulting in a sum and product that are each $2n$. When $n = 24$, the sum and product is $\boxed{48}$.

[NOTE: Other solutions sometimes exist, but no other solutions exist for $n = 2, 3, 4, 6, 24, 114$, or 174.]

Problem 4-6



The Angle Bisector Theorem tells us that the bisector of an angle of a triangle splits the opposite side in proportion to the other two sides. The vertical altitude of the isosceles triangle has length 8 and is split by the angle bisector in the same ratio as the other two sides of the left-side triangle $= (4+6):(4+2) = 10:6 = 5:3$, as shown. Point P, whose distance from two sides is 2, is shown. The longer legs of the small right triangles are 4, as shown. Here's why: at the lower left, the right triangle with shorter leg 3 and longer leg 6 is similar to the right triangle with shorter leg 2, so the latter triangle's longer leg is 4, and the overall base of length 12 is split into segments of length 4, 2, and 6. By the Pythagorean Theorem, the sum of the squares of the distances from point P to the 3 vertices of the large triangle is $(2^2+6^2) + (2^2+4^2) + (2^2+8^2) = \boxed{128}$.

Contest # 5 — *Answers & Solutions* — 2/11/20

Problem 5-1

The solutions are -1, -2, and -3. The sum of their reciprocals is $-1 - \frac{1}{2} - \frac{1}{3} = \boxed{\frac{-11}{6}}$.

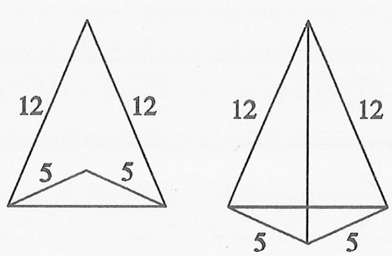
Problem 5-2

In any triangle, the length of the longest side must be less than the sum of the lengths of the other two sides, so $2020+n < 2020-n + 2020$, and $n < 1010$. The largest possible integer value of n is $\boxed{1009}$ (and such a triangle does exist).

Problem 5-3

This is basically a pigeon hole question. If there are 100 students and the only mode is 0, then we must make sure that no score occurs more often than a score of 0. Divide 100 by 16 (the number of possible scores) to get 6 (remainder 4). Thus, if each score is earned at most 7 times, we'd have 4 scores of 7 and 12 scores of 6. This doesn't allow for only one mode. If one score occurs 8 times, 2 scores occur 7 times, and the remaining 13 scores occur 6 times, we'd have accounted for all 100 students, and the only mode would be the score that occurred $\boxed{8}$ times.

Problem 5-4



The vertex angles of the two isosceles triangles are supplementary, so the 4 base angles of the isosceles triangles have a sum of 180°. Reflecting the smaller isosceles triangle across the common base creates two congruent right triangles with legs of length 5 and 12. The two altitudes together make up the common hypotenuse of the two right triangles. Its length is $\boxed{13}$.

Problem 5-5

Clearly, the numbers in question must have 3 digits each. Call one of them ABC. Call the other CBA. Since $A \times C$ ends in a 5, one of the numbers A or C—let's say its A—must be 5. Since $92{,}565 \div 500 < 200$, $C = 1$. To determine the value of B we notice that the 6 in the product is the last digit of $5B+B = 6B$, so $B = 1$ or $B = 6$. All that remains is to test the 2 possibilities, $\boxed{165 \ \& \ 561}$ (which work).

Problem 5-6

In a geometric sequence with common ratio r and first term a, the first 3 terms are a, ar, and ar^2. We are told that $a+4$, ar, and ar^2 form an arithmetic sequence, so $r \neq 1$ and $ar^2 - ar = ar-(a+4) = ar-a-4$. This leads to $a(r^2-2r+1) = -4$, so $a = \frac{-4}{(r-1)^2}$. Since a and r are integers, $(r-1)^2$ is a positive divisor of -4. The equation $(r-1)^2 = 2$ has no rational solution, so the non-zero solutions of $(r-1)^2 = 1$ or 4 are the only possibilities. From these two equations, we find that $r = 2$, 3, or -1. The three corresponding (a,b,c) triples are: $\boxed{(-4,-8,-16),\ (-1,-3,-9),\ (-1,1,-1)}$.

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Contest # 6 — *Answers & Solutions* — 3/17/20

Problem 6-1

Since 99999/11 leaves a remainder of 9, the largest 5-digit integer divisible by 11 is $99999 - 9 = \boxed{99990}$.

Problem 6-2

What is the largest perfect square less than 2020? Since $\sqrt{2020} \approx 44.9$, the largest perfect square less than 2020 is $44^2 = 1936$. Therefore, $2020 - n = 1936$, and $n = \boxed{84}$.

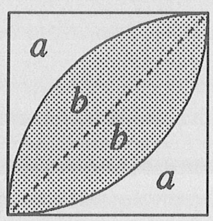
Problem 6-3

Of the first 8 tosses, exactly 1 will be "heads"; and the ninth toss must also be "heads." Only 8 permutations of the first 8 tosses contain 1 "heads" (7 "tails" and 1 "heads" can differ only in which toss is "heads"). The probability is $8\times(\frac{1}{2})^1(\frac{1}{2})^7(\frac{1}{2}) = \frac{1}{2^6} = \boxed{\frac{1}{64}}$.

Problem 6-4

Method I:

From the area of 2 quarter-circles (each with radius 2 and area $a+2b$), subtract the area of the square, $2a+2b$. The result is $2b$, the shaded region's area. Its area is $\pi+\pi-4 = \boxed{2\pi-4}$.



Method II:

A quarter-circle $(a+2b)$ minus a right triangle $(a+b)$ is $b = \pi-2$. Double that to get $2b = 2\pi-4$, the area of the shaded region.

Problem 6-5

If $Y = 200y$ & $X = 200x$, then $Y = 200y = 200x^2 = 200\left(\frac{X}{200}\right)^2 = \frac{1}{200}X^2$, so $a = \boxed{\frac{1}{200}}$.

Problem 6-6

If $x = \log_4 a = \log_{10} b = \log_{25}(a+b)$, we can conclude that $4^x = a$, $10^x = b$ and $25^x = a+b$. It follows that $4^x+10^x = 25^x = a+b$, and $\frac{a}{b} = \frac{4^x}{10^x} = \left(\frac{2}{5}\right)^x$. Also, $\frac{b}{a+b} = \frac{10^x}{25^x} = \left(\frac{2}{5}\right)^x$. Therefore, $\frac{a}{b} = \frac{b}{a+b}$, or $\frac{b}{a} = \frac{a}{b}+1$. Now, if we let $c = \frac{a}{b}$, then we get that $\frac{1}{c} = c+1$. The positive value of $c = \frac{a}{b}$ is the positive solution of $c^2+c-1 = 0$. That value of $c = \frac{a}{b}$ is $\boxed{\frac{-1+\sqrt{5}}{2}}$.

[**NOTE:** The value of x is $\log_{2/5}(c)$, which, to seven decimal places, is 0.5251737 . . . .]

