Contest # 4 *Answers & Solutions* 1/5/21

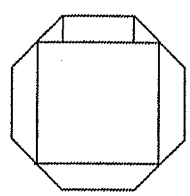
Problem 4-1

We want to add two different primes, but we don't want the sum to be a third prime. Let's add 2 small primes until the sum is composite (but remember that 1 is not a prime): $2+3=5$; $2+5=7$; $3+5=\boxed{8}$.

Problem 4-2

The hypotenuse could be either y or $4^2=16$. If the hypotenuse is 16 and one leg is $3^2=9$, then $y^2=$ (other leg)$^2=16^2-9^2=256-81=175$. If 4^2 is a leg, then $y^2=$ (hypotenuse)$^2=(3^2)^2+(4^2)^2=81+256=337$, so y could be $\boxed{\sqrt{175},\ \sqrt{337}}$.

Problem 4-3



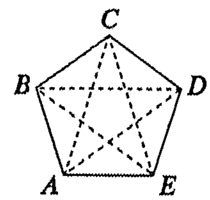
Perpendiculars to the square from vertices of any side of the octagon not touched by the square form 2 isosceles rt. triangles, each with hypotenuse-length $\sqrt{2}$ (half a side of the octagon). The isos. rt. triangles' legs have length 1, so the square's side is $1+2\sqrt{2}+1=2+2\sqrt{2}$. The perimeter of the square is $\boxed{4(2+2\sqrt{2}) \text{ or } 19.31}$.

Problem 4-4

Since $2^{11}=2048$ and $2^4=16$, $2^{11}-2^n<2021$ for $n=10, 9, 8, 7, 6$, or 5. That's 6 cases. All the other cases are of the form 2^m-2^n for $m=10, 9, \ldots, 1$ and $0 \le n \le m-1$: 10 values n for $m=10$, 9 values n for $m=9, \ldots, 1$ value of n for $m=1$. In addition to the first 6 cases noted above, the number of further cases is $10+9+\ldots+2+1=55$. The total number of positive integers that can be written as a difference of powers of 2 is $55+6=\boxed{61}$.

[NOTE: For integers $a>b\ge 0$ and $m>n\ge 0$, we have $2^m-2^n=2^a-2^b$ if and only if $m=a$ and $n=b$.]

Problem 4-5



Use a regular n-gon as a model, each person at a different vertex. Two people are friends/strangers if they are at vertices connected with a solid/dashed line segment. Three people are friends/strangers if they're at vertices of a triangle with 3 solid/dashed sides. The 5-person pentagon seen at the right shows that, with only 5 people, there need not be 3 friends or 3 strangers since there is no triangle which has 3 solid sides or which has 3 dashed sides each of whose vertices is 1 of the 5 points A, B, C, D, E. The diagram also works for 3 or 4 people. What if there are 6 people? Draw all 15 line segments (some solid, some dashed) that join the 6 points pairwise. Of the 5 segments from any vertex (say, A), at least 3 (say B, C, D) must be the same style, say solid. If any segment connecting B, C, and D is solid, we'd have a solid triangle (with those 2 vertices and A). If not, then $\triangle BCD$ has 3 dashed sides, so the answer is $\boxed{6}$.

[NOTE: Ramsey's Theorem generalizes this problem.]

Problem 4-6

For the maximum product, the tens digits must be 9, 8, 7, 6, and 5. For positive numbers with a fixed sum, as their differences decrease, their product increases. In other words, to maximize the product of n numbers, bunch them up closely to their average. Minimize the differences to get $\boxed{90, 81, 72, 63, 54}$.

Here's a formal proof: In any such 2-digit number, the tens digit will be larger than the units digit (otherwise switching them will increase the product). Represent the 2-digit number $90+b$ (with tens digit 9 and ones digit b) as $9b$. We'll show that $b=0$. Suppose $b \ne 0$. In a maximal product, let the 2-digit number ending in 0 be $a0=10a+0$. Since $9b \times a0=(90+b)(10a+0)=900a+10ab<900a+90b=(90+0)(10a+b)=90\times AB$ (with AB the 2-digit number $10a+b$), we showed that $9b\times a0<90\times AB$. For maximal product, we must have $9b=90$; so one number is 90. Look at the number with tens digit 8. By an argument like above (but more involved), the ones digit is 1. And so on for the rest. From all this, we get the 5 numbers above.