Key

Calculus AB Course Review: Unit 4 Antidifferentiation MC WS

Integral Methods Priority Order Checklist:

1) **Expand/Power Rule** $ex: \int \frac{(x^2-2)^2}{\sqrt{x}}dx$
*only one term in the denominator
*Always check **first** to see if problem can be expanded BEFORE attempting U-substitution

2) **U-Substitution:** $ex: \int \frac{3x}{(x^2-2)^2}dx$
*u-value is expression inside parentheses
*rewrite problem using parentheses to identify u-value
*Most Integral Problems fall in this category

3) **U-Sub/Change of Variable:** $\int 5x\sqrt{3-x}dx$
*initial u-value not enough to remove x
*re-arrange assigned u-value equation to solve for x in terms of u

4) **Long/Synthetic Division** $\int \frac{3x^2-2x+1}{x-4}dx$
*condition: numerator degree is same or higher than denominator

5) **Arc-Trig U-Sub:** $\int \frac{4}{\sqrt{1-x^2}}dx$
*Condition: Denominator degree is 2 or more degrees greater than numerator
condition applies for terms in rational form, not trig or exponential terms

6) 6) **Arc-Trig U-Sub:** $\int \frac{4x}{x^4-4x^2+19}dx$
*Condition: Denominator degree is 2 or more degrees greater than numerator (*for rational expressions*)
* Complete the square in denominator to match Arc-Trig Integral Rules

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1) $\int \left(x - \frac{1}{2x}\right)^2 dx =$ expand $\left(x-\frac{1}{2x}\right)\left(x-\frac{1}{2x}\right)$

$x^2 - \frac{1}{2} - \frac{1}{2} + \frac{1}{4x^2}$

$\int x^2 + \frac{1}{4}x^{-2} - 1\, dx = \frac{x^3}{3} + \frac{1}{4}\left(\frac{x^{-1}}{-1}\right) - x + C$

(A) $\frac{1}{3}\left(x - \frac{1}{2x}\right)^3 + C$ (B) $x^2 - 1 + \frac{1}{4x^2} + C$ (C) $\frac{x^3}{3} - 2x - \frac{1}{4x} + C$

(D) $\frac{x^3}{3} - x - \frac{4}{x} + C$ $\boxed{\text{(E)}}$ none of these $\boxed{\frac{x^3}{3} - \frac{1}{4x} - x + C}$

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cannot expand
apply u-sub

u-sub

2) $\int \frac{1-3y}{\sqrt{2y-3y^2}}dy = \int \frac{1-3y}{(2y-3y^2)^{1/2}}dy \rightarrow \int (1-3y)(2y-3y^2)^{-1/2}dy$

$u = 2y-3y^2$
$\frac{du}{dy} = 2-6y$

$dy = \frac{du}{2(1-3y)}$

(A) $4\sqrt{2y-3y^2} + C$ (B) $\frac{1}{4}(2y-3y^2)^2 + C$ (C) $\frac{1}{2}\ln\sqrt{2y-3y^2} + C$

(D) $\frac{1}{4}(2y-3y^2)^{1/2} + C$ $\boxed{\text{(E)}}$ $\sqrt{2y-3y^2} + C$

$\int \frac{\cancel{(1-3y)}u^{-1/2}du}{2\cancel{(1-3y)}} \rightarrow \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + c \rightarrow u^{1/2} + C = \boxed{\sqrt{2y-3y^2} + C}$

cannot expand, use u-sub

3) $\int \frac{x\,dx}{1+4x^2} =$ $u=1+4x^2$, $\frac{du}{dx}=8x$ | $dx=\frac{du}{8x}$, $\int \frac{\cancel{x}}{u}\cdot\frac{du}{8\cancel{x}}$ | $\frac{1}{8}\int \frac{1}{u}\,du = \frac{1}{8}\ln|u|+C$

$\boxed{\text{(A)}}$ $\frac{1}{8}\ln(1+4x^2)+C$ (B) $\frac{1}{8(1+4x^2)^2}+C$ (C) $\frac{1}{4}\sqrt{1+4x^2}+C$

(D) $\frac{1}{2}\ln|1+4x^2|+C$ (E) $\frac{1}{2}\tan^{-1}2x+C$

$\boxed{\frac{1}{8}\ln|1+4x^2|+C}$

u-sub, not an expansion problem

4) $\int \frac{x}{(1+4x^2)^2}\,dx =$ $u=1+4x^2$, $\frac{du}{dx}=8x$ | $dx=\frac{du}{8x}$, $\int \frac{\cancel{x}}{u^2}\cdot\frac{du}{8\cancel{x}}$ | $\frac{1}{8}\int u^{-2}\,du = \frac{-1}{8}\left(\frac{u^{-1}}{-1}\right)+C$

(A) $\frac{1}{8}\ln(1+4x^2)^2+C$ (B) $\frac{1}{4}\sqrt{1+4x^2}+C$ $\boxed{\text{(C)}}$ $-\frac{1}{8(1+4x^2)}+C$

(D) $-\frac{1}{3(1+4x^2)^3}+C$ (E) $-\frac{1}{(1+4x^2)}+C$

$\frac{-1}{8u}+C$

$\boxed{\frac{-1}{8(1+4x^2)}+C}$

not expandable, not u-sub
*denominator variable 2 degrees greater than numerator

5) $\int \frac{dy}{\sqrt{4-y^2}} =$

(A) $\frac{1}{2}\sin^{-1}\frac{y}{2}+C$ (B) $-\sqrt{4-y^2}+C$ $\boxed{\text{(C)}\ \sin^{-1}\frac{y}{2}+C}$

(D) $-\frac{1}{2}\ln\sqrt{4-y^2}+C$ (E) $-\frac{1}{3(4-y^2)^{3/2}}+C$

* $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$

$\int \frac{du}{\sqrt{(2)^2-(y)^2}}$ $a=2$, $u=y$, $du=dy$

$\boxed{\arcsin\left(\frac{y}{2}\right)+C \text{ or } \sin^{-1}\left(\frac{y}{2}\right)+C}$

6) $\int \frac{2x+1}{2x}\,dx =$ expand $\frac{1}{2}\int \frac{2x+1}{x} \rightarrow \frac{1}{2}\int (2x+1)x^{-1}\,dx = \frac{1}{2}\int 2 + \frac{1}{x}\,dx$

$= \frac{1}{2}\left(2x + \ln|x|\right) + C$

$= \boxed{x + \frac{1}{2}\ln|x| + C}$

(A) $\boxed{x + \frac{1}{2}\ln|x| + C}$ (B) $1 + \frac{1}{2}x^{-1} + C$ (C) $x + 2\ln|x| + C$

(D) $x + \ln|2x| + C$ (E) $\frac{1}{2}\left(2x - \frac{1}{x^2}\right) + C$

7) $\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} =$ cannot expand, apply u-sub

$\int \frac{\cos x}{(1+\sin x)^{1/2}}\,dx$ $u = 1+\sin x$, $\frac{du}{dx} = \cos x$ | $dx = \frac{du}{\cos x}$

$\int \frac{\cancel{\cos x}}{u^{1/2}} \cdot \frac{du}{\cancel{\cos x}}$ | $\frac{u^{1/2}}{1/2} + c$

$\int u^{-1/2}\,du$

$\boxed{2(1+\sin x)^{1/2} + c}$

or

$\boxed{2\sqrt{1+\sin x} + c}$

(A) $-\frac{1}{2}(1+\sin x)^{1/2} + C$

(B) $\ln\sqrt{1+\sin x} + C$

(C) $\boxed{2\sqrt{1+\sin x} + C}$

(D) $\ln|1+\sin x| + C$

(E) $\frac{2}{3(1+\sin x)^{3/2}} + C$

8) $\int \sec\left(\frac{t}{2}\right) dt =$ u-sub

$u = \frac{t}{2} = \frac{1}{2}t$, $\frac{du}{dt} = \frac{1}{2}$ | $dt = 2du$ | $\int \sec u \cdot 2du = 2\int \sec u\,du$

(A) $\ln\left|\sec\frac{t}{2} + \tan\frac{t}{2}\right| + C$ (B) $2\tan^2\frac{t}{2} + C$ (C) $2\ln\cos\frac{t}{2} + C$

(D) $\ln|\sec t + \tan t| + C$ (E) $\boxed{2\ln\left|\sec\frac{t}{2} + \tan\frac{t}{2}\right| + C}$

* $\int \sec u\,du = \ln|\sec u + \tan u| + c$

$2\int \sec u\,du = 2\ln|\sec u + \tan u| + c$

$\boxed{2\ln\left|\sec\left(\frac{t}{2}\right) + \tan\left(\frac{t}{2}\right)\right| + c}$

denominator variable is 2 degrees greater than numerator → Arctan, complete square

* complete the square:

x^2+2x+2

$x^2+2x+1+2-1 \quad \left(\frac{b}{2}\right)^2$

9) $\int \frac{dx}{x^2+2x+2} =$

(A) $\ln(x^2+2x+2)+C$ (B) $\ln|x+1|+C$ $\boxed{\text{(C)}}$ $\arctan(x+1)+C$

(D) $\frac{1}{\frac{1}{3}x^3+x^2+2x}+C$ (E) $-\frac{1}{x}+\frac{1}{2}\ln|x|+\frac{x}{2}+C$

$\int \frac{dx}{(x+1)^2+1}$

$\int \frac{dx}{(x+1)^2+(1)^2}$

* $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

$u=x+1 \quad \frac{du}{dx}=1$

$a=1 \quad dx=du$

$\hookrightarrow \frac{1}{1}\arctan\left(\frac{x+1}{1}\right)+C$

$=\boxed{\arctan(x+1)+C}$

10) $\int \frac{(2-y)^2}{4\sqrt{y}}\,dy =$ *expand *power Rule

(A) $\frac{1}{6}(2-y)^3\sqrt{y}+C$

(B) $2\sqrt{y}-\frac{2}{3}y^{3/2}+\frac{8}{5}y^{5/2}+C$

(C) $\ln|y|-y+2y^2+C$

$\boxed{\text{(D)}}$ $2y^{1/2}-\frac{2}{3}y^{3/2}+\frac{1}{10}y^{5/2}+C$

(E) none of these

$\frac{1}{4}\int (2-y)^2 y^{-1/2}\,dy$

$\frac{1}{4}\int (4-4y+y^2)y^{-1/2}\,dy$

$\frac{1}{4}\int 4y^{-1/2}-4y^{1/2}+y^{3/2}\,dy$

$\frac{1}{4}\left(4\left(\frac{y^{1/2}}{1/2}\right)-4\left(\frac{y^{3/2}}{3/2}\right)+\frac{y^{5/2}}{5/2}+C\right)$

$\boxed{2y^{1/2}-\frac{2}{3}y^{3/2}+\frac{1}{10}y^{5/2}+C}$

* $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

*cannot expand *not u-sub * Arctan Rule

11) $\int \frac{e^x}{1+e^{2x}}\,dx =$

$\int \frac{e^x}{(1)^2+(e^x)^2}\,dx$

$a=1 \quad u=e^x \quad \frac{du}{dx}=e^x \quad dx=\frac{du}{e^x}$

$\boxed{\text{(A)}}$ $\tan^{-1}e^x+C$ (B) $\frac{1}{2}\ln(1+e^{2x})+C$ (C) $\ln(1+e^{2x})+C$

(D) $\frac{1}{2}\tan^{-1}e^x+C$ (E) $2\tan^{-1}e^x+C$

$\int \frac{\cancel{e^x}}{a^2+u^2}\cdot\frac{du}{\cancel{e^x}} = \boxed{\frac{1}{1}\arctan\left(\frac{e^x}{1}\right)+C \text{ or } \tan^{-1}(e^x)+C}$

Definite Integrals and Applications:

12) $\int_1^2 \frac{x^2+6x+6}{x+1}\,dx =$

cannot expand not u-sub *degree in numerator is greater than denominator *Apply Long/synthetic division

(A) $1+\ln\frac{3}{2}$

(B) 6.5

$\boxed{\text{(C)}}$ $6.5+\ln\frac{3}{2}$

(D) $6.5+\ln 6$

$$
\begin{array}{r}
x+5+\frac{1}{x+1} \\
x+1\,\overline{)\,x^2+6x+6} \\
-x^2-x \\
\hline
5x+6 \\
-5x-5 \\
\hline
1
\end{array}
$$

$\int x+5+\frac{1}{x+1}\,dx \quad u=x+1 \quad \int \frac{1}{u}du = \ln|x+1|$

$\left.\frac{x^2}{2}+5x+\ln|x+1|\right]_1^2$

$\frac{2^2}{2}+5(2)+\ln|2+1|-\left(\frac{1}{2}+5+\ln|2|\right)$

$12+\ln 3-5.5-\ln 2$

$=\boxed{6.5+\ln\left(\frac{3}{2}\right)}$

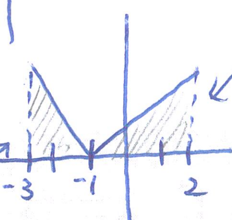
$\ln a-\ln b=\ln\left(\frac{a}{b}\right)$

$\ln 3-\ln 2=\ln\left(\frac{3}{2}\right)$

Definite Integrals and Applications:

sketch graph
Add areas of triangles
Area $= \frac{1}{2}bh$

$y = |x+1|$



$\frac{1}{2}(3)(3)$

$\frac{1}{2}(2)(2)$

13) $\displaystyle\int_{-3}^{2} |x+1|\,dx =$

(A) $\frac{5}{2}$ (B) $\frac{7}{2}$ (C) 5 (D) $\frac{11}{2}$ $\boxed{\text{(E)}}$ $\frac{13}{2}$

$$\frac{1}{2}(2)(2) + \frac{1}{2}(3)(3)$$
$$= \frac{4}{2} + \frac{9}{2} = \boxed{\frac{13}{2}}$$

14) $\displaystyle\frac{d}{dt}\int_0^t \sqrt{x^3+1}\,dx =$ *Apply SFTC $\displaystyle\frac{d}{dx}\int_a^{p(x)} f(t)\,dt = f(p(x))\cdot p'(x)$

$\boxed{\text{(A)}}$ $\sqrt{t^3+1}$ (B) $\dfrac{\sqrt{t^3+1}}{3t^2}$ (C) $\frac{2}{3}(t^3+1)(\sqrt{t^3+1}-1)$

(D) $3x^2\sqrt{x^3+1}$ (E) none of these

$$\sqrt{t^3+1}\cdot 1 = \boxed{\sqrt{t^3+1}}$$

15) $\displaystyle\frac{d}{dx}\int_{\pi/2}^{x^2} \sqrt{\sin t}\,dt =$ Apply SFTC $\displaystyle\frac{d}{dx}\int_a^{p(x)} f(t)\,dt = f(p(x))\cdot p'(x)$

$$\longrightarrow = \sqrt{\sin(x^2)}\cdot 2x = \boxed{2x\sqrt{\sin x^2}}$$

(A) $\sqrt{\sin t^2}$ (B) $2x\sqrt{\sin x^2}-1$ (C) $\frac{2}{3}(\sin^{3/2} x^2 - 1)$

(D) $\sqrt{\sin x^2}-1$ $\boxed{\text{(E)}}$ $2x\sqrt{\sin x^2}$

u-sub

16) $\displaystyle\int_0^1 xe^{x^2}\,dx =$ $\displaystyle\int x e^{x^2}\,dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$\int \cancel{x} e^u \cdot \frac{du}{2\cancel{x}} = \frac{1}{2}e^u$$

$$\frac{1}{2}e^{x^2}\Big]_0^1 = \frac{1}{2}e^1 - \frac{1}{2}e^0$$

(A) $e-1$ $\boxed{\text{(B)}}$ $\frac{1}{2}(e-1)$ (C) $2(e-1)$ (D) $\frac{e}{2}$ (E) $\frac{e}{2}-1$

$$\boxed{\frac{1}{2}e - \frac{1}{2}(1) \text{ or } = \frac{1}{2}(e-1)}$$

17) $\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta =$ $\int (\cos\theta)^2 \sin\theta \, d\theta$ u-sub

$u = \cos\theta$
$\frac{du}{d\theta} = -\sin\theta$
$d\theta = \frac{du}{-\sin\theta}$

(A) $-\frac{2}{3}$ (B) $\frac{1}{3}$ (C) 1 $\boxed{\text{(D)}}$ $\frac{2}{3}$ (E) 0

$\int u^2 \cancel{\sin\theta} \cdot \frac{du}{-\cancel{\sin\theta}}$

$-\int u^2 du$

$-\frac{u^3}{3}$

$-\frac{1}{3}(\cos\theta)^3$

$-\frac{1}{3}(\cos\theta)^3\Big]_0^{\pi} = -\frac{1}{3}(\cos\pi)^3 - \left(-\frac{1}{3}(\cos 0)^3\right)$

$= -\frac{1}{3}(-1)^3 + \frac{1}{3}(1)^3$

$= \frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$

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18) $\int_0^1 \frac{e^{-x}+1}{e^{-x}} dx =$ $\int (e^{-x}+1)e^{x} dx$

expand first

$\boxed{\text{(A)}}$ e (B) $2+e$ (C) $\frac{1}{e}$ (D) $1+e$ (E) $e-1$

$\int e^0 + e^x dx$

$\int 1 + e^x dx$

$x + e^x\Big]_0^1 = 1 + e^1 - (0 + e^0)$

$= \cancel{1} + e - 0 - \cancel{1}$

$= \boxed{e}$

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19) If the substitution $u = \sqrt{x+1}$ is used, then $\int_0^3 \frac{dx}{x\sqrt{x+1}}$ is equivalent to

(change of variable)
*u-substitution
*convert bounds to be in terms of u

(A) $\int_1^2 \frac{du}{u^2-1}$ $\boxed{\text{(B)}}$ $\int_1^2 \frac{2\,du}{u^2-1}$ (C) $2\int_0^3 \frac{du}{(u-1)(u+1)}$

(D) $2\int_1^2 \frac{du}{u(u^2-1)}$ (E) $2\int_0^3 \frac{du}{u(u-1)}$

$u = \sqrt{x+1} = (x+1)^{1/2}$

$\frac{du}{dx} = \frac{1}{2}(x+1)^{-1/2}(1)$

$\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$

$dx = 2\sqrt{x+1}\, du$

$\int \frac{1}{x \cdot \cancel{\sqrt{x+1}}} \cdot 2\cancel{\sqrt{x+1}}\, du$

$2\int \frac{du}{x}$

$* u = \sqrt{x+1}$

$u^2 = (\sqrt{x+1})^2$

$u^2 = x+1$

$u^2 - 1 = x$

$2\int \frac{du}{u^2-1}$

*convert bounds
If $x=0$, $u=\sqrt{0+1}=1$
If $x=3$, $u=\sqrt{3+1}=2$

$2\int_1^2 \frac{du}{u^2-1} \rightarrow \boxed{\int_1^2 \frac{2du}{u^2-1}}$