Key

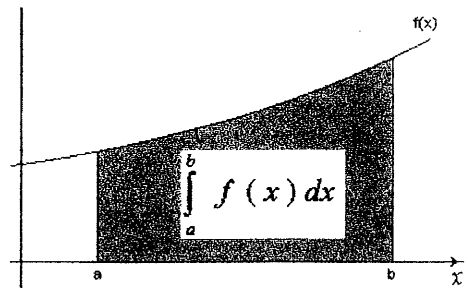
1

Calculus **Chapters 4.3/4.4a** **1st Fundamental Theorem of Calculus**

$* \int_a^b f'(x)\,dx = f(b) - f(a)$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative of f.



Recall:

*The general derivative is a **slope-finding function** or formula : (ex. $f'(x) = 2x + 1$)

*The specific derivative is the **actual slope** at a point (ex: $f'(3) = 7$)

Likewise…

The indefinite integral is an **Area-Finding Function** or formula (Ex: $\int 2x dx = x^2 + C$)

The definite integral is the **Actual Area** of the region for an interval (Ex: $\int_1^3 2x dx = 8$)

*If a function is **continuous** on a closed interval, then the function is able to be integrated on that interval

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Class Examples:

1. Evaluate $\int_1^4 (3x^2 + 4x - 1)dx$

$$\frac{3x^3}{3} + \frac{4x^2}{2} - x \Big]_1^4$$

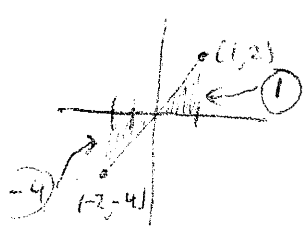
$$x^3 + 2x^2 - x \Big]_1^4$$

$$4^3 + 2(4)^2 - 4 - (1^3 + 2 - 1)$$

$$92 - 2 = \boxed{90}$$

For definite integrals:
**NOTE: we don't need to worry about the constant of integration " +C ". It will always wash out.

2. Evaluate $\int_{-2}^{1} 2xdx = \frac{2x^2}{2} = x^2\Big]_{-2}^{1} = 1^2 - (-2)^2 = \boxed{-3}$



portions of graph below x-axis will result in negative value.

2

Integral Properties:

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (given that c is between a and b)

Example 3: If $\int_0^3 f(x)dx = 4$ and $\int_3^6 f(x)dx = -1$, find the below:

a) $\int_0^6 f(x)dx = \int_0^3 f(x)dx + \int_3^6 f(x)dx = 4 + (-1) = \boxed{3}$

b) $\int_6^3 f(x)dx = -\int_3^6 f(x)dx = -(-1) = \boxed{1}$

c) $\int_3^3 f(x)dx = \boxed{0}$

d) $\int_3^6 (-5f(x)+3)dx = -5\int_3^6 f(x)dx + \int_3^6 3\,dx$

$3x\Big]_3^6 = 18 - 9 = 9$

$= -5(-1) + 9 = \boxed{14}$

Ex. 4: If $\int_3^8 f'(x)dx = 10$ and $f(8) = 6$, find $f(3)$.

*Reminder that the FFTC can be used as an equation solving tool to find the value of an antiderivative at a specific point. Recall that $\int_a^b f'(x)dx = f(b) - f(a)$

$\int_3^8 f'(x)dx = f(8) - f(3)$

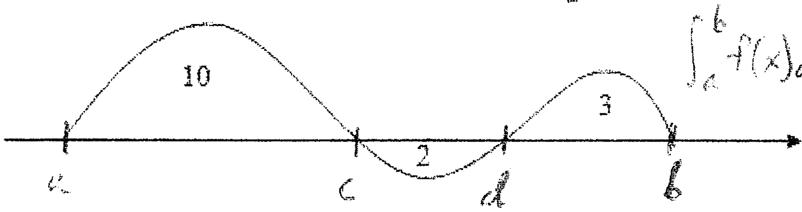
$10 = 6 - f(3)$

$10 - 6 = -f(3)$

$4 = -f(3)$

$\boxed{f(3) = -4}$

Ex. 5: The area for each region is given. Find $\int_a^b f(x)dx$



$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx$

$= 10 + (-2) + 3$

$= \boxed{11}$

Non-AP Calculus Ch. 4.3-4.4 Definite Integrals Classwork Problems

41. Using Properties of Definite Integrals Given

$$\int_0^5 f(x)\,dx = 10 \quad \text{and} \quad \int_5^7 f(x)\,dx = 3$$

$\int_5^0 f(x)dx = -10$

evaluate

(a) $\int_0^7 f(x)\,dx.$

$\int_0^5 f(x) + \int_5^7 f(x)dx$

$10 + 3 = \boxed{13}$

(b) $\int_5^0 f(x)\,dx.$

$= \boxed{-10}$

(c) $\int_5^5 f(x)\,dx.$

$\boxed{0}$

(d) $\int_0^5 3f(x)\,dx.$

$3\int_0^5 f(x)dx$

$= 3(10) = \boxed{30}$

42. Using Properties of Definite Integrals Given

$$\int_0^3 f(x)\,dx = 4 \quad \text{and} \quad \int_3^6 f(x)\,dx = -1$$

evaluate

(a) $\int_0^6 f(x)\,dx.$

$\int_0^3 f(x)dx + \int_3^6 f(x)dx$

$4 + -1 = \boxed{3}$

(b) $\int_6^3 f(x)\,dx.$ $= \boxed{1}$

(c) $\int_3^3 f(x)\,dx.$

$= \boxed{0}$

(d) $\int_3^6 -5f(x)\,dx.$

$-5\int_3^6 f(x)dx$

$= -5(-1) = \boxed{5}$

43. Using Properties of Definite Integrals Given

$$\int_2^6 f(x)\,dx = 10 \quad \text{and} \quad \int_2^6 g(x)\,dx = -2$$

evaluate

(a) $\int_2^6 [f(x) + g(x)]\,dx.$

$\int_2^6 f(x)dx + \int_2^6 g(x)dx$

$= 10 + -2 = \boxed{8}$

(b) $\int_2^6 [g(x) - f(x)]\,dx.$

$\int_2^6 g(x)dx - \int_2^6 f(x)dx$

$= 10 - (-2) = \boxed{12}$

(c) $\int_2^6 2g(x)\,dx.$

$2\int_2^6 g(x)dx$

$2(-2) = \boxed{-4}$

(d) $\int_2^6 3f(x)\,dx.$

$3\int_2^6 f(x)dx$

$3(10) = \boxed{30}$

49. Think About It Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ $\neq \int_0^5 f(x)dx + 2$

$\int_0^5 f(x)dx + \int_0^5 2dx$

$2x\Big]_0^5 = 2(5) - 2(0)$

$= 10$

$4 + 10 = \boxed{14}$

④

Evaluating a Definite Integral In Exercises 5–34, evaluate the definite integral. Use a graphing utility to verify your result.

8. $\int_{-1}^{2} (7 - 3t)\, dt$

$7t - \frac{3t^2}{2}\Big]_{-1}^{2} = 7(2) - \frac{3(2)^2}{2} - \left(7(-1) - \frac{3(-1)^2}{2}\right)$

$= 14 - 6 - (-7 - \frac{3}{2})$

$= 8 - (-8.5) = \boxed{16.5 = \frac{33}{2}}$

10. $\int_{1}^{2} (6x^2 - 3x)\, dx = \frac{6x^3}{3} - \frac{3x^2}{2}\Big]_{1}^{2}$

$2x^3 - \frac{3}{2}x^2\Big]_{1}^{2} = 2(2)^3 - \frac{3}{2}(2)^2 - \left(2(1)^3 - \frac{3}{2}(1)^2\right)$

$= 16 - 6 - (2 - \frac{3}{2}) = 9.5$

$= \boxed{\frac{19}{2}}$

13. $\int_{1}^{2} \left(\frac{3}{x^2} - 1\right) dx$

$\int 3x^{-2} - 1\, dx$

$\frac{3x^{-1}}{-1} - x\Big]_{1}^{2}$

$-\frac{3}{x} - x\Big]_{1}^{2} = \frac{-3}{2} - 2 - (-3 - 1) = \boxed{\frac{1}{2}}$

16. $\int_{-8}^{8} x^{1/3}\, dx = \frac{x^{4/3}}{4/3}\Big]_{-8}^{8}$ $+3/3$

$\frac{3}{4}x^{4/3}\Big]_{-8}^{8} = \frac{3}{4}(8)^{4/3} - \frac{3}{4}(-8)^{4/3}$

$= \frac{3}{4}(16) - \frac{3}{4}(-16) =$

$12 + 12 = \boxed{24}$

17. $\int_{-1}^{1} (\sqrt[3]{t} - 2)\, dt$

$\int t^{1/3} - 2\, dt$

$\frac{t^{4/3}}{4/3} - 2t$

$\frac{3}{4}t^{4/3} - 2t\Big]_{-1}^{1} = \frac{3}{4}(1) - 2(1) - \left[\frac{3}{4}(-1)^{4/3} - 2(-1)\right]$

$\frac{3}{4} - 2 - \frac{3}{4} - 2 = \boxed{-4}$

18. $\int_{1}^{8} \sqrt{\frac{2}{x}}\, dx$ $\int \sqrt{2}\, x^{-1/2}\, dx$

$\sqrt{2} \cdot \frac{x^{1/2}}{1/2} = 2\sqrt{2}x^{1/2}\Big]_{1}^{8}$

$= 2\sqrt{2}(8)^{1/2} - 2\sqrt{2}(1)^{1/2}$

$= 2\sqrt{2}(\sqrt{8}) - 2\sqrt{2}$

$= (2\sqrt{2})(2\sqrt{2}) - 2\sqrt{2}$

$= \boxed{4(2) - 2\sqrt{2} \approx 5.172}$

Key

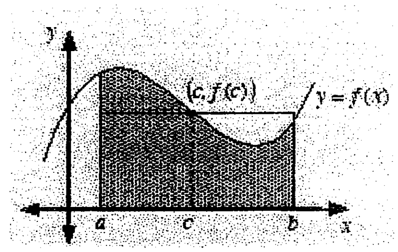
Calculus **Chapter 4.4b** **Average Value Theorem (MVT for Integrals)**

If function f is integrable on the closed interval [a, b], then the average value of f on the interval is

Area

$$f(c)=\frac{1}{b-a}\left[\int_a^b f(x)dx\right]$$

height width



Since height · width = Area , height = $\frac{\text{Area}}{\text{width}}$

*There exists a rectangle such that the area of the rectangle is the same as the area under the curve (shaded region). f (c) is the height of the rectangle

Example 1: a) Find the average value of $f(x) = x^2 + 1$ on [2, 5]. b) find the c value

* Use Avg. Value Theorem : $f(c)=\frac{1}{5-2}\int_2^5 x^2+1\,dx$

a) $f(c)=\frac{1}{3}\cdot\left.\frac{x^3}{3}+x\right]_2^5 = \frac{1}{3}\left[\frac{5^3}{3}+5-\left(\frac{2^3}{3}+2\right)\right]$

$= \frac{1}{3}\left[\frac{117}{3}+3\right] = \frac{1}{3}\left(\frac{126}{3}\right) = 14$ $\boxed{f(c)=14}$

b) Find c-value

$f(x) = x^2+1$

$f(c) = c^2+1$

$14 = c^2+1$

$13 = c^2$

$c = \pm\sqrt{13}$

$\boxed{c=\sqrt{13} \text{ since } 2<\sqrt{13}<5}$

6

Key 6

47. *Think About It* The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

$A_T = \frac{1}{2}bh$

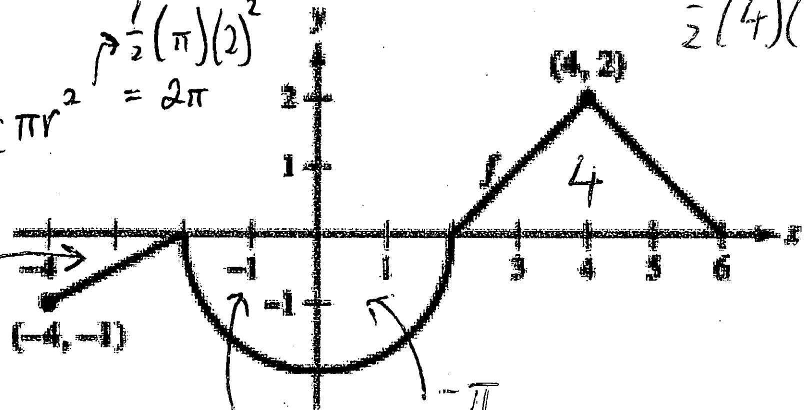
$A_{semicircle}: \frac{1}{2}\pi r^2$

$\frac{1}{2}(\pi)(2)^2 = 2\pi$

$\frac{1}{2}(4)(2)$

$\frac{1}{2}(2)(1)$

-1



(a) $\int_0^2 f(x)\,dx = \boxed{-\pi}$ (b) $\int_2^6 f(x)\,dx = \boxed{4}$

(c) $\int_{-4}^2 f(x)\,dx \; \boxed{-2\pi-1}$ (d) $\int_{-4}^6 f(x)\,dx \; \boxed{3-2\pi}$

(e) $\int_{-4}^6 |f(x)|\,dx$ (f) $\int_{-4}^6 [f(x)+2]\,dx$

$\boxed{2\pi+5}$

$$\int_{-4}^{6} f(x)\,dx + \int_{-4}^{6} 2\,dx$$

$$2x\Big]_{-4}^{6}$$

$12-(-8)$

$3-2\pi + 20$

$\boxed{23-2\pi}$

Chapter 4.4b Average Value Theorem Classwork Problems

Finding the Average Value of a Function In Exercises 51–56, find the average value of the function over the given interval and all values of x in the interval for which the function equals its average value.

Avg. value:
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

51. $f(x) = 9 - x^2$, $[-3, 3]$

$$f(c) = \frac{1}{3-(-3)}\int_{-3}^{3} 9 - x^2\,dx$$

$$\frac{1}{6} \cdot 9x - \frac{x^3}{3}\Big]_{-3}^{3} = 9(3) - \frac{(+3)^3}{3} - \left[9(-3) - \frac{(-3)^3}{3}\right]$$

$$\frac{1}{6}(36) = \boxed{6}$$

a) $\boxed{f(c) = 6}$

$9 - x^2 = 6$

b) $\boxed{x = \pm\sqrt{3}}$

52. $f(x) = \dfrac{4(x^2 + 1)}{x^2}$, $[1, 3]$ $= (4x^2+4)x^{-2} = 4 + 4x^{-2}$

$$f(c) = \frac{1}{3-1}\int_1^3 4 + 4x^{-2}\,dx$$

$$\frac{1}{2} \cdot 4x + \frac{4x^{-1}}{-1}\Big]_1^3 = 4(3) - \frac{4}{3} - (4-4)$$

$$= \frac{1}{2}\left(\frac{32}{3}\right) = \frac{16}{3}$$

a) $\boxed{f(c) = \frac{16}{3}}$

$\dfrac{4(x^2+1)}{x^2} = \dfrac{16}{3}$

b) $\boxed{x = \sqrt{3}}$

53. $f(x) = x^3$, $[0, 1]$

$$f(c) = \frac{1}{1-0}\int_0^1 x^3 = 1 \cdot \frac{x^4}{4}\Big]_0^1 = \frac{1}{4} - 0$$

a) $\boxed{f(c) = \frac{1}{4}}$

$x^3 = \frac{1}{4}$

b) $\boxed{x = \sqrt[3]{1/4} \approx 0.63}$

54. $f(x) = 4x^3 - 3x^2$, $[0, 1]$

$$f(c) = \frac{1}{1-0}\int_0^1 4x^3 - 3x^2\,dx = \frac{4x^4}{4} - \frac{3x^3}{3}\Big]_0^1 = 1 - 1 = 0$$

a) $\boxed{f(c) = 0}$

b) $4x^3 - 3x^2 = 0$

$x^2(4x - 3) = 0$

$\boxed{x = 0,\ x = 3/4}$

55. $f(x) = \sin x$, $[0, \pi]$

$$f(c) = \frac{1}{\pi - 0}\int_0^\pi \sin x = \frac{1}{\pi} \cdot (-\cos x)\Big]_0^\pi$$

$$\frac{1}{\pi}(-\cos \pi) - \frac{1}{\pi}(-\cos 0)$$

$$\frac{1}{\pi}(1) - \frac{1}{\pi}(-1) = \frac{2}{\pi}$$

a) $\boxed{f(c) = \frac{2}{\pi}}$

b) $\sin x = \frac{2}{\pi}$

$x = \sin^{-1}\left(\frac{2}{\pi}\right) \approx \boxed{0.881}$

56. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$

$$f(c) = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x\,dx \quad \frac{2}{\pi} \cdot \sin x\Big]_0^{\pi/2} = \sin(\pi/2) - \sin(0)$$

a) $f(c) = \frac{2}{\pi}$

$\cos x = \frac{2}{\pi}$

b) $x = \cos^{-1}\left(\frac{2}{\pi}\right) \approx \boxed{0.881}$

8

4.5a U-Substitution Classwork Problems

Finding an Indefinite Integral In Exercises 5–26, find the indefinite integral and check the result by differentiation.

6. $\int (x^2-9)^3(2x)\,dx$

$u = x^2 - 9$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$\int u^3 \cdot \cancel{2x} \cdot \frac{du}{\cancel{2x}} = \frac{u^4}{4} + C$

$= \boxed{\frac{1}{4}(x^2-9)^4 + C}$

7. $\int \sqrt{25-x^2}\,(-2x)\,dx = \int (25-x^2)^{1/2}(-2x)\,dx$

$u = 25 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$\int u^{1/2}(\cancel{-2x}) \cdot \frac{du}{\cancel{-2x}} = \int u^{1/2}\,du$

$= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(25-x^2)^{3/2} + C}$

8. $\int \sqrt[3]{3-4x^2}\,(-8x)\,dx$

$\int (3-4x^2)^{1/3}(-8x)\,dx$

$u = 3 - 4x^2$

$\frac{du}{dx} = -8x$

$du = -8x\,dx$

$\frac{du}{-8x} = dx$

$\int u^{1/3} \cdot \cancel{-8x} \cdot \frac{du}{\cancel{-8x}} = \int u^{1/3}\,du$

$= \frac{u^{4/3}}{4/3} + C = \frac{3}{4}u^{4/3} + C$

$= \boxed{\frac{3}{4}(3-4x^2)^{4/3} + C}$

15. $\int 5x\,\sqrt[3]{1-x^2}\,dx$ $\quad \int 5x(1-x^2)^{1/3}\,dx$

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$\int 5\cancel{x} \cdot u^{1/3} \cdot \frac{du}{-2\cancel{x}} = -\frac{5}{2}\int u^{1/3}\,du$

$-\frac{5}{2} \cdot \frac{u^{4/3}}{4/3} = -\frac{5}{2} \cdot \frac{3}{4}u^{4/3} + C$

$= \boxed{-\frac{15}{8}(1-x^2)^{4/3} + C}$

20. $\int \frac{6x^2}{(4x^3-9)^3}\,dx$

$u = 4x^3 - 9$

$\frac{du}{dx} = 12x^2$

$du = 12x^2\,dx$

$\frac{du}{12x^2} = dx$

$\int \frac{6\cancel{x^2}}{u^3} \cdot \frac{du}{12\cancel{x^2}} = \int \frac{6}{12} \cdot \frac{1}{u^3}\,du$

$\frac{1}{2}\int u^{-3}\,du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C$

$-\frac{1}{4u^2} + C = \boxed{\frac{-1}{4(4x^3-9)^2} + C}$

22. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$ $\quad \int \frac{x^3}{(1+x^4)^{1/2}}\,dx$

$\int x^3(1+x^4)^{-1/2}\,dx$

$u = 1 + x^4$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{du}{4x^3} = dx$

$\int \cancel{x^3} \cdot u^{-1/2} \cdot \frac{du}{4\cancel{x^3}}$

$\frac{1}{4}\int u^{-1/2}\,du$

$\frac{1}{4} \cdot \frac{u^{1/2}}{1/2} + C$

$\frac{1}{4} \cdot \frac{2}{1}u^{1/2} + C$

$\boxed{\frac{1}{2}(1+x^4)^{1/2} + C}$

9

23. $\int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt$

$u = 1 + t^{-1}$

$\frac{du}{dt} = -t^{-2}$

$\frac{du}{dt} = \frac{-1}{t^2}$

$-dt = t^2 du$

$dt = -t^2 du$

$\int u^3 \cdot \left(\frac{1}{\cancel{t^2}}\right) \cdot -\cancel{t^2} du$

$= -\int u^3 du$

$= \frac{-u^4}{4} + C$

$-\frac{1}{4}u^4 + C$

$= \boxed{-\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C}$

24. $\int \left[x^2 + \frac{1}{(3x)^2}\right] dx$

$\int x^2 + \frac{1}{9x^2} dx$

$\int x^2 + \frac{1}{9}x^{-2} dx$

$\frac{x^3}{3} + \frac{1}{9} \cdot \frac{x^{-1}}{-1} + C$

$\boxed{\frac{1}{3}x^3 - \frac{1}{9x} + C}$

33. $\int \pi \sin \pi x \, dx$

$\int \pi \sin(\pi x)\, dx$

$u = \pi x$

$\frac{du}{dx} = \pi$

$du = \pi dx$

$\frac{du}{\pi} = dx$

$\int \cancel{\pi} \sin u \cdot \frac{du}{\cancel{\pi}}$

$\int \sin u\, du = -\cos u + C$

$= \boxed{-\cos(\pi x) + C}$

36. $\int \csc^2\left(\frac{x}{2}\right) dx$

$u = \frac{x}{2} = \frac{1}{2}x$

$\frac{du}{dx} = \frac{1}{2}$

$dx = 2du$

$\int \csc^2 u \cdot 2du = 2\int \csc^2 u\, du$

$= 2(-\cot u) + C$

$= \boxed{-2\cot\left(\frac{x}{2}\right) + C}$

37. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta}\, d\theta$

$\int \frac{1}{\theta^2} \cos\left(\frac{1}{\theta}\right) d\theta$

$u = \frac{1}{\theta} = \theta^{-1}$

$\frac{du}{d\theta} = -\theta^{-2}$

$\frac{du}{d\theta} = \frac{-1}{\theta^2}$

$-d\theta = \theta^2 du$

$d\theta = -\theta^2 du$

$\int \frac{1}{\cancel{\theta^2}} \cos u \cdot -\cancel{\theta^2} d\theta$

$-\int \cos u\, du$

$= -\sin u + C$

$= \boxed{-\sin\left(\frac{1}{\theta}\right) + C}$

40. $\int \sqrt{\tan x} \sec^2 x\, dx$

$\int (\tan x)^{1/2} \cdot \sec^2 x\, dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\, dx$

$\frac{du}{\sec^2 x} = dx$

$\int u^{1/2} \cdot \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}}$

$= \frac{u^{3/2}}{3/2} + C$

$= \boxed{\frac{2}{3}(\tan x)^{3/2} + C}$

41. $\int \frac{\csc^2 x}{\cot^3 x} dx$

$\int \frac{(\csc x)^2}{(\cot x)^3} dx$

$u = \cot x$

$\frac{du}{dx} = -\csc^2 x$

$du = -\csc^2 x\, dx$

$\frac{du}{-\csc^2 x} = dx$

$\int \frac{\cancel{\csc^2 x}}{u^3} \cdot \frac{du}{-\cancel{\csc^2 x}}$

$= -\int u^{-3} du = -\frac{u^{-2}}{-2} + C$

$= \frac{1}{2u^2} + C$

$\boxed{= \frac{1}{2(\cot x)^2} + C}$

42. $\int \frac{\sin x}{\cos^3 x} dx$

$\int \sin x \cdot (\cos x)^{-3} dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\, dx$

$\frac{du}{-\sin x} = dx$

$\int \cancel{\sin x} \cdot u^{-3} \cdot \frac{du}{-\cancel{\sin x}}$

$-\int u^{-3} du = \frac{-u^{-2}}{-2} + C$

$\frac{1}{2u^2} + C$

$= \boxed{\frac{1}{2(\cos x)^2} + C}$

10

4.5b U-Substitution Change of Variable

Change of Variables In Exercises 47–54, find the indefinite integral by the method shown in Example 5.

Assigned relationship between x and u

47. $\int x\sqrt{x+6}\,dx$ $\qquad \int x(x+6)^{1/2}\,dx$

$u = x+6$

$\frac{du}{dx} = 1$

$dx = du$

$\int x \cdot u^{1/2} \cdot du$

$\to u - 6 = x$

$\int (u-6)u^{1/2}\,du$

$\int u^{1/2}(u-6)\,du$

$\int u^{3/2} - 6u^{1/2}\,du$

$\frac{u^{5/2}}{5/2} - 6\frac{u^{3/2}}{3/2} + C$

$\frac{2}{5}u^{5/2} - 6\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{\frac{2}{5}(x+6)^{5/2} - 4(x+6)^{3/2} + C}$

48. $\int x\sqrt{3x-4}\,dx$ $\qquad \int x(3x-4)^{1/2}\,dx$

$u = 3x-4$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{du}{3} = dx$

$\int x \cdot u^{1/2} \cdot \frac{du}{3}$

$\to u+4 = 3x$

$\frac{u+4}{3} = x$

$\int \frac{u+4}{3} \cdot u^{1/2} \cdot \frac{du}{3}$

$\frac{1}{9}\int u^{1/2}(u+4)\,du$

$\frac{1}{9}\int u^{3/2} + 4u^{1/2}\,du$

$\int \frac{1}{9}u^{3/2} + \frac{4}{9}u^{1/2}\,du$

$\frac{1}{9}\cdot\frac{u^{5/2}}{5/2} + \frac{4}{9}\cdot\frac{u^{3/2}}{3/2} + C$

$\frac{1}{9}\cdot\frac{2}{5}u^{5/2} + \frac{4}{9}\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{\frac{2}{45}(3x-4)^{5/2} + \frac{8}{27}(3x-4)^{3/2} + C}$

49. $\int x^2\sqrt{1-x}\,dx$ $\qquad \int x^2(1-x)^{1/2}\,dx$

$u = 1-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x^2 \cdot u^{1/2} \cdot -du$

$\to x = 1-u$

$\int (1-u)^2 \cdot u^{1/2} \cdot -du$

$-\int u^{1/2}(1-u)^2\,du$

$= -\int u^{1/2}(1-2u+u^2)\,du$

$= \int -u^{1/2} + 2u^{3/2} - u^{5/2}\,du$

$\frac{-u^{3/2}}{3/2} + \frac{2u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + C$

$-\frac{2}{3}u^{3/2} + 2\cdot\frac{2}{5}u^{5/2} - \frac{2}{7}u^{7/2} + C$

$\boxed{-\frac{2}{3}(1-x)^{3/2} + \frac{4}{5}(1-x)^{5/2} - \frac{2}{7}(1-x)^{7/2} + C}$

50. $\int (x+1)\sqrt{2-x}\,dx$ $\qquad \int (x+1)(2-x)^{1/2}\,dx$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x+1 \cdot u^{1/2} \cdot -du$

$\to x = 2-u$

$\int (2-u+1)u^{1/2}(-du)$

$-\int u^{1/2}(3-u)\,du$

$\int -3u^{1/2} + u^{3/2}\,du$

$-\frac{3u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2} + C$

$-3\cdot\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$\boxed{= -2(2-x)^{3/2} + \frac{2}{5}(2-x)^{5/2} + C}$

Calculus **Chapter 4.5a** **U-Substitution Method for Indefinite Integrals**

Key

11

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

*U-substitution is a method of rewriting an integral problem into a simpler one to help us identify an Integral Rule appropriate for the problem.

Example 1:

Suppose $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$
$f'(x) = 3\cos(3x)$

This means that:

$$\int 3\cos(3x)dx = \sin(3x) + C$$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int x \cdot u^{15} \cdot \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{15} du = \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Be sure that variable x's cancel out. Remaining constants, coefficients are ok.

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$\frac{1}{6}\int \sec^2 u\, du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

Ex. 4: $\int x^3\sqrt{5-x^4}\,dx = \int x^3(5-x^4)^{1/2}dx$

$u = 5-x^4$

$\frac{du}{dx} = -4x^3$

$dx = \frac{du}{-4x^3}$

$\int \cancel{x^3} \cdot u^{1/2} \cdot \frac{du}{-4\cancel{x^3}}$

$-\frac{1}{4}\int u^{1/2} du$

$= -\frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C$

$= \boxed{-\frac{1}{6}(5-x^4)^{3/2} + C}$

(12)

Ex. 5: $\int \tan^5 x \sec^2 x dx$

$\int (\tan x)^5 (\sec x)^2 dx$

$u = \tan x$ | $dx = \frac{du}{\sec^2 x}$

$\frac{du}{dx} = \sec^2 x$

$\int (u)^5 \cdot \sec^2 x \cdot \frac{du}{\sec^2 x} = \int u^5 du$

$= \frac{u^6}{6} + C = \boxed{\frac{1}{6}\tan^6 x + C}$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right)dy$

$\int (3-y)(y^{-1/2})dy$

$\int 3y^{-1/2} - y^{1/2} dy$

$\frac{3y^{1/2}}{1/2} - \frac{y^{3/2}}{3/2} + C$

$\boxed{6y^{1/2} - \frac{2}{3}y^{3/2} + C}$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}dx$

$\int x(x+3)^{1/2}dx$

$u = x+3$

$\frac{du}{dx} = 1$

$dx = du$

*Creative method of substitution in order to eliminate x-variable

$\int x \cdot u^{1/2} du$

$x = u - 3$

$\int (u-3)u^{1/2} du$

$\int u^{3/2} - 3u^{1/2} du$

$\frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2} + C$

$\boxed{\frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C}$

Ex. 8: $\int x^2\sqrt{2-x}dx$

$\int x^2(2-x)^{1/2}dx$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x^2 \cdot u^{1/2} \cdot (-du)$

$x = 2-u$

$-\int (2-u)^2 u^{1/2} du$

$-\int (4-4u+u^2)u^{1/2} du$

$= \int -4u^{1/2} + 4u^{3/2} - u^{5/2}$

$= -\frac{4u^{3/2}}{3/2} + \frac{4u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + C$

$\boxed{-\frac{8}{3}(2-x)^{3/2} + \frac{8}{5}(2-x)^{5/2} - \frac{2}{7}(2-x)^{7/2} + C}$

Key 13

Calculus **Chapter 4.5b** **U-Substitution Method for Definite Integrals**

U-Substitution with definite integrals: Be sure the bounds matches the variable of the problem

Ex. 1: $\int_1^2 2x(x^2-2)^3\,dx$

$u = x^2 - 2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \cancel{2x}\cdot u^3 \cdot \frac{du}{\cancel{2x}}$

$\int u^3\,du$

Convert bounds:

if $x=1$, $u = 1^2 - 2 = -1$

if $x=2$, $u = 2^2 - 2 = 2$

$\int_{-1}^{2} u^3\,du$

$= \frac{u^4}{4}\Big]_{-1}^{2} = \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right) = \frac{16}{4} - \frac{1}{4}$

$= \boxed{\frac{15}{4}}$

OR:

$\int u^3\,du = \frac{u^4}{4} = \frac{(x^2-2)^4}{4}\Big]_{1}^{2}$

$= \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$

Ex. 2: $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{x}{u^{1/2}} \cdot \frac{du}{2}$

* Need to use change of variable method:

$u = 2x - 1$

$\frac{u+1}{2} = x$

$\int \frac{\frac{u+1}{2}}{u^{1/2}} \cdot \frac{du}{2}$

$\frac{1}{4}\int (u+1)u^{-1/2}\,du$

$\frac{1}{4}\int u^{1/2} + u^{-1/2}\,du$

If $x=1$, $u = 2(1) - 1 = 1$

if $x=5$, $u = 2(5) - 1 = 9$

$\frac{1}{4}\frac{u^{3/2}}{3/2} + \frac{1}{4}\frac{u^{1/2}}{1/2}$

$\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}\Big]_1^9 = \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right)$

$= \frac{1}{6}(27) + \frac{1}{2}(3) - \frac{1}{6} - \frac{1}{2}$

$= \boxed{\frac{16}{3}}$

OR $\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}$

$= \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2}\Big]_1^5$

$= \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right) = \boxed{\frac{16}{3}}$

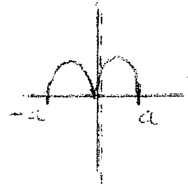
Integrals of Odd and Even Functions

Review: Suppose $\int_{10}^{3} f(x)dx = 9$ and $\int_{-1}^{3} f(x)dx = 5$, find $\int_{-1}^{10} f(x)dx$

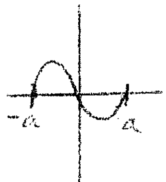
$$\int_{-1}^{10} f(x)dx = \int_{-1}^{3} f(x)dx + \int_{3}^{10} f(x)dx = 5 + (-9) = \boxed{-4}$$

Even/Odd Rules:

Even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

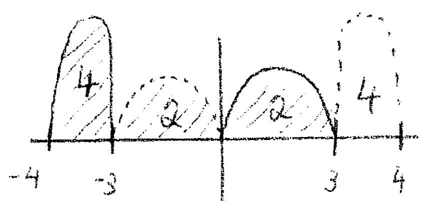


Odd: $\int_{-a}^{a} f(x)dx = 0$



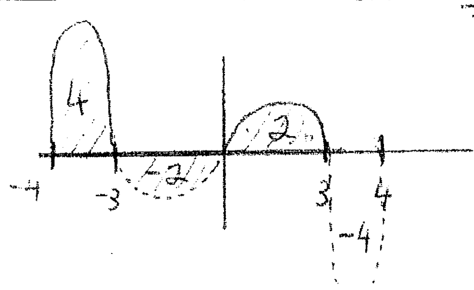
Ex. 3: Suppose g(x) is an even function where $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.

(Sketch a possible graph using the above given information)



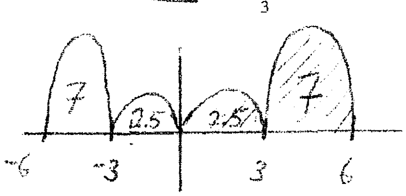
$$\int_{-4}^{3} g(x)dx = 4 + 2 + 2 = \boxed{8}$$

Ex. 4: Same as Example 3, but g(x) is an odd function: $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.



$$\int_{-4}^{3} g(x)dx = \boxed{4}$$

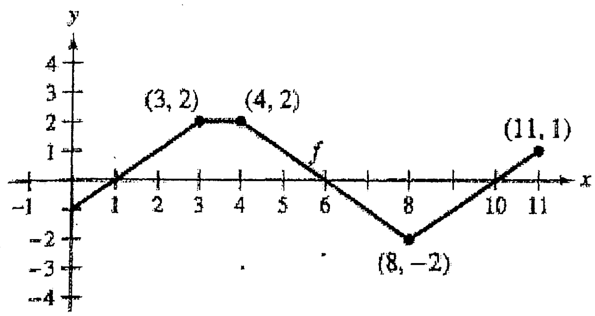
Ex. 5: If f(x) is even and $\int_{3}^{6} f(x)dx = 7$ and $\int_{-6}^{3} f(x)dx = 12$, find $\int_{0}^{6} f(x)dx$



$$\int_{0}^{6} f(x)dx = 2.5 + 7 = \boxed{9.5}$$

Ch. 4.3-4.4 Definite Integrals Selected Homework

Think About It The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx$

(b) $\int_3^4 3f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$

(d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$

(f) $\int_4^{10} f(x)\,dx$

Think About It Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$

(b) $\int_{-2}^3 f(x + 2)\,dx$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even.)

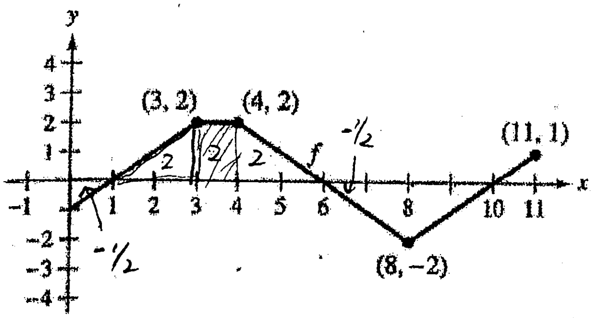
(d) $\int_{-5}^5 f(x)\,dx$ (f is odd.)

-16

Ch. 4.3-4.4 Definite Integrals Selected Homework

4.3

48) **Think About It** The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx = -(-\tfrac{1}{2}) = \boxed{\tfrac{1}{2}}$

(b) $\int_3^4 3f(x)\,dx \quad 3\int_3^4 f(x)\,dx = 3(2) = \boxed{6}$

(c) $\int_0^7 f(x)\,dx \quad 6 - 1 = \boxed{5}$

(d) $\int_5^{11} f(x)\,dx \quad -4 + 1 = \boxed{-3}$

(e) $\int_0^{11} f(x)\,dx \quad -\tfrac{1}{2} + 6 - 4 + \tfrac{1}{2} = \boxed{2}$

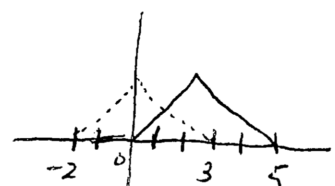
(f) $\int_4^{10} f(x)\,dx \quad 2 - 4 = \boxed{-2}$

49. **Think About It** Consider the function f that is continuous on the interval $[-5, 5]$ and for which

$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

$2x\Big]_0^5 = 2(5) - 2(0) = 10$



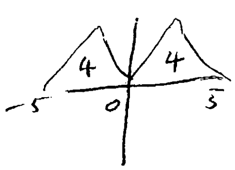
(a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx$
$= 4 + 10 = \boxed{14}$

(b) $\int_{-2}^{3} f(x+2)\,dx = \int_0^5 f(u)\,du = \boxed{4}$

$u = x + 2 \quad \frac{du}{dx} = 1$

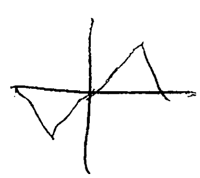
convert bounds:
$x = -2, \; u = x + 2 = -2 + 2 = 0$
$x = 3, \; u = x + 2 = 3 + 2 = 5$

(c) $\int_{-5}^{5} f(x)\,dx$ (f is even.)



$= \boxed{8}$

(d) $\int_{-5}^{5} f(x)\,dx$ (f is odd.)



$= \boxed{0}$