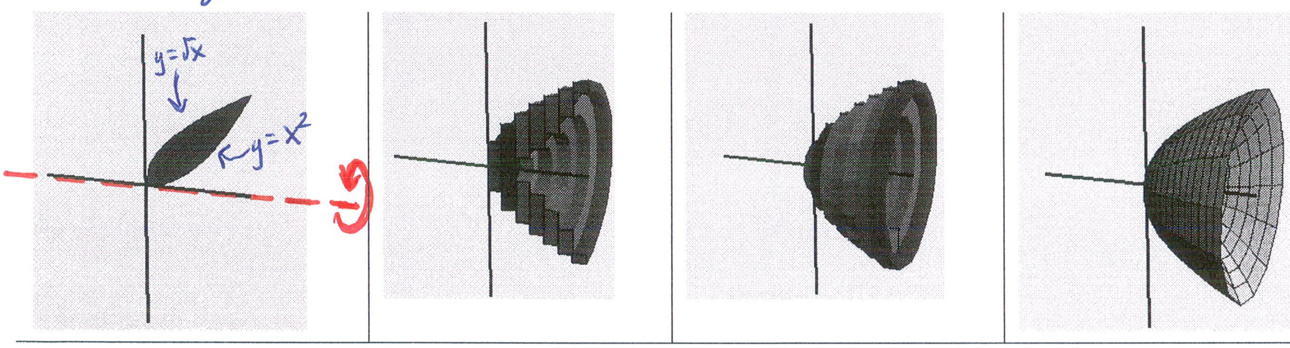
Calculus Ch. 7.2b: Volume by Washer Method

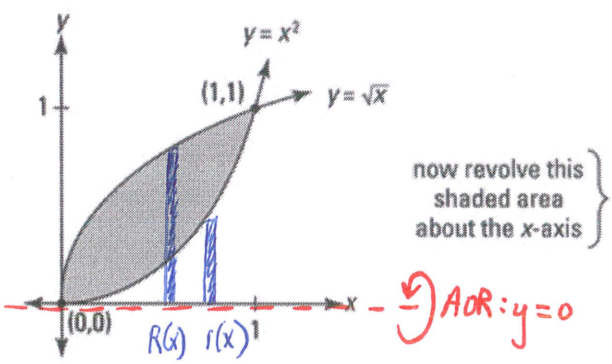
Key

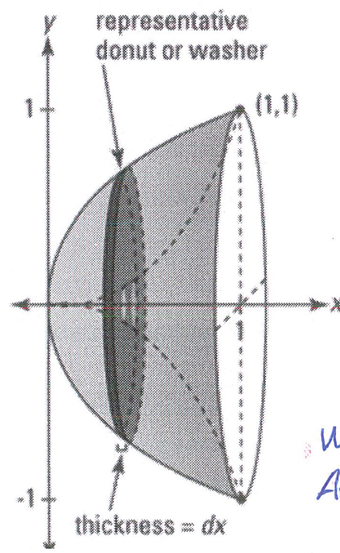
With **Disc Method**, we rotated one function around the x-axis. We used the Integral Notation to add areas of circular discs to find the volume of 3-dimensional curved objects.

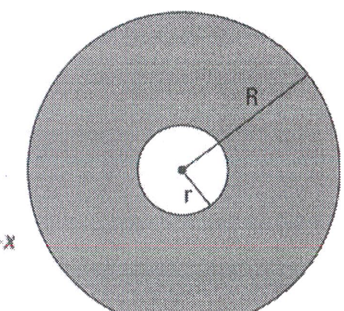
Now, what if we wanted to find the volume created between 2 functions?

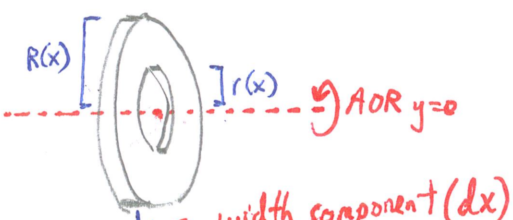
Take a look at the region between $y = x^2$ and $y = \sqrt{x}$. Picture taking that region and rotate that shape 360° around the x-axis. What shape do you see? What's different between this object and the object created by Disc Method? Washer Method objects are hollowed out like a bowl unlike a full compact solid











Washer Area = Outer Circle Area − Inner Circle Area

$\pi R^2 - \pi r^2$

Each slice has the shape of a washer (circular rings) (or donut) so its area equals the area of the entire circle minus the area of the hole. Area of circular washer (ring)= $\pi R^2 - \pi r^2$

$\underbrace{\pi[R^2 - r^2]}_{\text{Area}} \cdot \underbrace{dx}_{\text{width}} = \text{Volume}$

$$\textit{Volume (Washer Method)}: V = \pi\int_a^b [R(x)]^2 - [r(x)]^2\,dx$$

Integral calculus: Accumulating and stacking circular washers (rings) to form full Volume of curved object.

Volume (Washer Method): $V = \pi \int_a^b [R(x)^2 - r(x)^2]dx$

Washer Method Steps:

1)Confirm gap exists between x-axis and the shaded region (gap indicates hole → suitable for washer method)

2) Draw dotted line across the line of rotation to indicate location of Axis of Revolution **(AOR)**

3) Draw the length of **Radius R(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to further boundary of shaded region $R(x) = Top - Bottom$

4) Draw the length of **radius r(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to closer boundary of shaded region $r(x) = top - bottom$

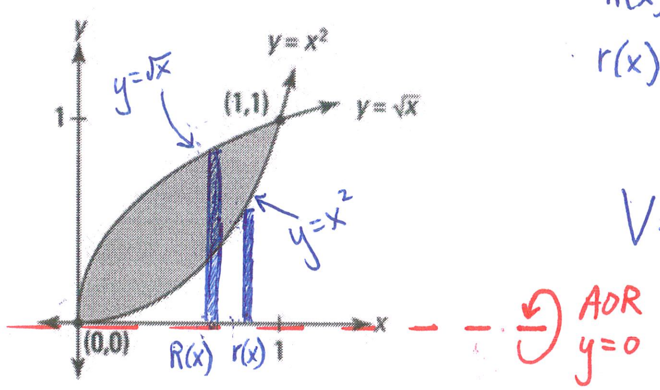
5) Identify the left and right bounds (a and b). If needed, set the equations equal to find bounds.

6) Enter expressions for R(x) and r(x) into Washer Method Volume formula

7) Enter Integral into calculator to find Volume. (TI-84: Math 9→ FnInt or TI-36X Pro: 2[nd] → e)

AOR: $y=0$

Example 1: Find the volume of the solid bounded by $y = x^2$ and $y = \sqrt{x}$ revolved about the x-axis.



$R(x) = \sqrt{x} - 0 = \sqrt{x}$

$r(x) = x^2 - 0 = x^2$

$V = \pi \int_a^b [R(x)]^2 - [r(x)]^2 dx$

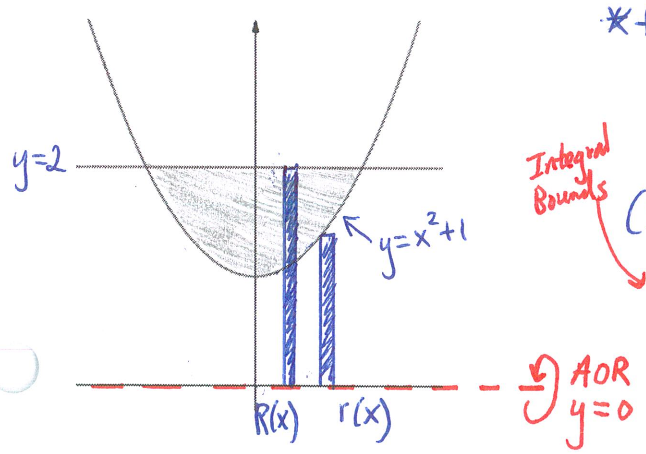
$V = \pi \int_0^1 [\sqrt{x}]^2 - [x^2]^2 dx$

$V = \pi \int_0^1 x - x^4 dx$

$\boxed{V = 0.3\pi \text{ or } \frac{3}{10}\pi \text{ units}^3}$

AOR: $y=0$

Example 2: Find the volume of the solid bounded by $y = x^2 + 1$ and $y = 2$ revolved about the x-axis.



*find bounds by setting equations equal:

$x^2 + 1 = 2$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

Integral Bounds

$x = 1, x = -1$

$R(x) = \underbrace{2}_{Top} - \underbrace{(0)}_{Bottom} = 2$

$r(x) = x^2 + 1 - (0) = x^2 + 1$

$V = \pi \int_{-1}^1 [R(x)]^2 - [r(x)]^2$

$V = \pi \int_{-1}^1 [2]^2 - [x^2+1]^2 dx$

$\boxed{V = \frac{64}{15}\pi \text{ units}^3}$

Washer Method Steps: $V = \pi \int_a^b [R(x)^2 - r(x)^2]dx$

1)Confirm gap exists between x-axis and the shaded region (gap indicates hole → suitable for washer method)

2) Draw dotted line across the AOR to indicate location of Axis of Revolution **(AOR)**

3) Draw the length of **Radius R(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to further boundary of shaded region

4) Draw the length of **radius r(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to closer boundary of shaded region

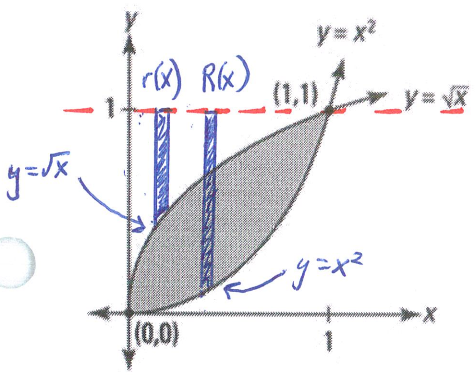
5) Identify the left and right bounds (a and b). If needed, set the equations equal to find bounds.

6) Enter expressions for R(x) and r(x) into Washer Method Volume formula

7) Enter Integral into calculator to find Volume. (TI-84: Math 9→ FnInt or TI-36X Pro: 2nd → e)

AOR: $y=1$

Example 3: Find the volume of the solid bounded by $y = x^2$ and $y = \sqrt{x}$ revolved about the line y = 1



AOR
$y=1$

Top − Bottom

$R(x) = 1 - x^2$

Top − Bottom

$r(x) = 1 - \sqrt{x}$

$$V = \pi \int_0^1 [1-x^2]^2 - [1-\sqrt{x}]^2 dx$$

$$\boxed{V = 0.3667\pi \text{ units}^3}$$

AOR: $y=4$

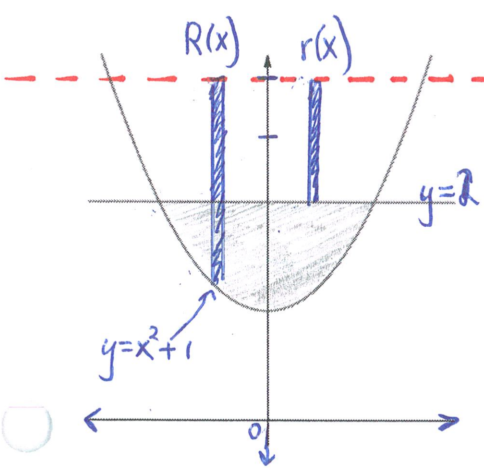
Example 4: Find the volume of the solid bounded by $y = x^2 + 1$ and $y = 2$ revolved about line y = 4

* find bounds by setting equations equal

$x^2+1=2$

$x^2-1=0$

$x=1, x=-1$ ← Integral bounds



AOR: $y=4$

Top − Bottom

$R(x) = 4 - (x^2+1)$

$R(x) = 4 - x^2 - 1$

$R(x) = 3 - x^2$

Top − Bottom

$r(x) = 4 - 2$

$r(x) = 2$

$$V = \pi \int_{-1}^1 [3-x^2]^2 - [2]^2 dx$$

$$\boxed{V = 6.4\pi \text{ or } \frac{32}{5}\pi \text{ units}^3}$$

Washer Method Steps: $V = \pi \int_a^b [R(x)^2 - r(x)^2]dx$

1)Confirm gap exists between x-axis and the shaded region (gap indicates hole → suitable for washer method)

2) Draw dotted line across the AOR to indicate location of Axis of Revolution **(AOR)**

[R(x) = Top − Bottom]

3) Draw the length of **Radius R(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to further boundary of shaded region

4) Draw the length of **radius r(x)**: Place pen/pencil **first** on the dotted line (AOR) and extend to closer boundary of shaded region

[r(x) = top − bottom]

5) Identify the left and right bounds (a and b). If needed, set the equations equal to find bounds.

6) Enter expressions for R(x) and r(x) into Washer Method Volume formula

7) Enter Integral into calculator to find Volume. (TI-84: Math 9→ FnInt or TI-36X Pro: 2nd → e)

AOR: $y = -3$

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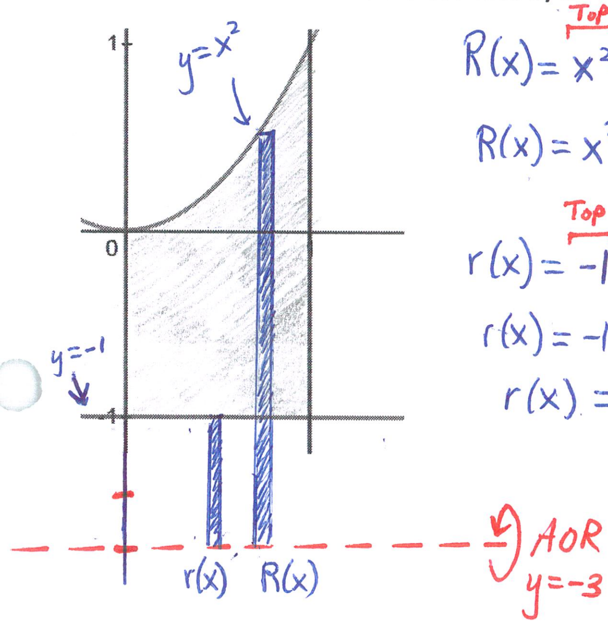
5. **Find the volume of the solid bounded by x = 1, y =- 1, y-axis, and the graph $y = x^2$ rotated about the line y = -3**



$R(x) = \underbrace{x^2}_{Top} - \underbrace{(-3)}_{Bottom}$

$R(x) = x^2 + 3$

$r(x) = \underbrace{-1}_{Top} - \underbrace{(-3)}_{Bottom}$

$r(x) = -1 + 3$

$r(x) = 2$

$$V = \pi \int_0^1 [x^2+3]^2 - [2]^2 dx$$

$$\boxed{V = 7.2\pi \text{ or } \frac{36}{5}\pi \text{ units}^3}$$

AOR: $y = 8$

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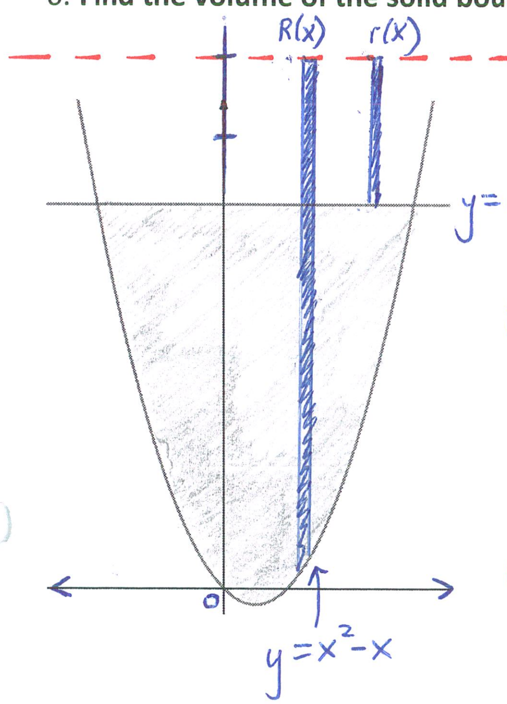
6. **Find the volume of the solid bounded by equations y = x^2 – x and y = 6 rotated about the line y = 8**



AOR: $y = 8$

*find bounds by setting equations equal

$x^2 - x = 6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, x = -2$ ← Integral bounds

$R(x) = \underbrace{8}_{Top} - \underbrace{(x^2 - x)}_{Bottom}$

$= 8 - x^2 + x$

$R(x) = 8 - x^2 + x$

$r(x) = \underbrace{8}_{top} - \underbrace{6}_{bottom}$

$r(x) = 2$

$$V = \pi \int_{-2}^{3} [8 - x^2 + x]^2 - [2]^2 dx$$

$$\boxed{V = 187.5\pi \text{ or } \frac{375}{2}\pi \text{ units}^3}$$