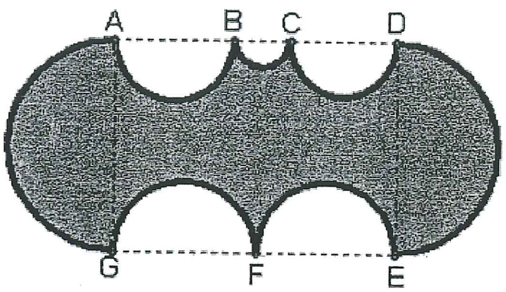
Milton Math Team Problems SOLUTIONS Oct. 1 **2015**

1.

The figure below is a rectangle with semicircular appendages and indentations. Given that AB = CD = 8, GF = FE = 10, AG = DE = 25, and BC = 4, what is the length of the border of the shaded region?



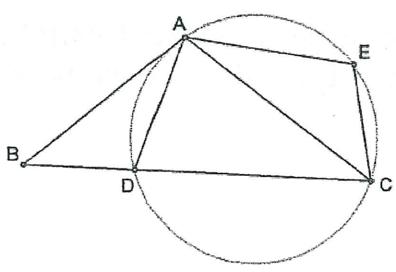
Solution:

Each of the semicircles has circumference of length $\pi/2$ times its diameter. Therefore the total length of the figure's boundary is

$$\frac{\pi}{2}(8+4+8+10+10+25+25) = \frac{\pi}{2}\cdot 90 = 45\pi.$$

2.

In the figure below, points A, D, C, and E all lie on the same circle, and D lies between points B and C. Furthermore, ∠BAD ≅ ∠CAE. If AE = 6, AC = 10 and AD = 9, what is the length of segment $\overline{AB}$?



By definition, angles BDA and DAC are supplementary.

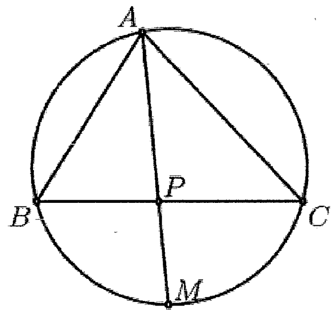
Also, since angles ADC and AEC inscribe the same segment in a circle from opposite sides, they are also supplementary. Since angles BDA and AEC share a common supplement, they are congruent. Since angles BAD and CAE are also congruent, triangles DAB and EAC are similar.

Since triangles DAB and EAC are similar, $AB/AC = AD/AE$,' or

$$AB = \frac{\cdot AC}{AE} = \frac{9\cdot 10}{6} = 15.$$

3.

Points A, B, C, and M lie on the same circle, as shown in the diagram below. Point M is the midpoint of the arc from B to C that does not contain A, and lines AM and BC intersect at P. Given that $AP \cdot PM = 12$ and $AB/AC = 3$, what is the length of BC?

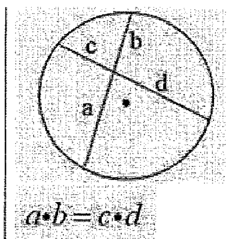


Recall:

Intersecting Chords Rule

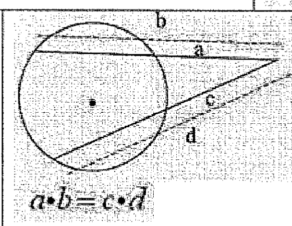
If two chords intersect in a circle, the product of the lengths of the segments of one chord equal the product of the segments of the other.

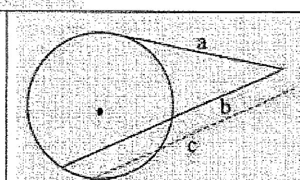
Part * Part = Part * Part



Secant and Tangent
(or Secant and Secant)

Outside * Whole = Outside * Whole





a * a = b * c

SOLUTION:

Since small arcs BM and MC are equal, it follows that angles BAM and MAC are equal; in other words, AM is the bisector of angle BAC. Thus by the angle bisector theorem,

$$BP/PC = AB/AC = 3.$$

On the other hand, by the Power of a Point theorem,

$$BP \cdot PC = AP \cdot PM = 12.$$

$$BP^2 = BP \cdot PC \cdot \frac{BP}{PC} = 12 \cdot 3 = 36,$$

so $BP = 6$. Similarly,

$$PC^2 = BP \cdot PC \cdot \frac{PC}{BP} = 12 \cdot \frac{1}{3} = 4,$$

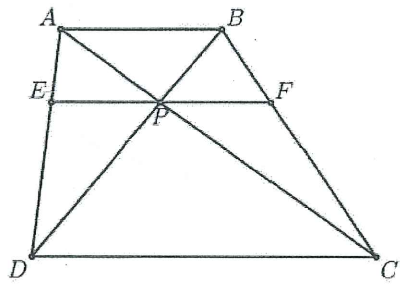
so $PC = 2$. Therefore $BC = BP + PC = 6 + 2 = 8$.

4.

In trapezoid $ABCD$, lines AB and DC are parallel. Points E and F lie on segments DA and CB, respectively, in such a way that

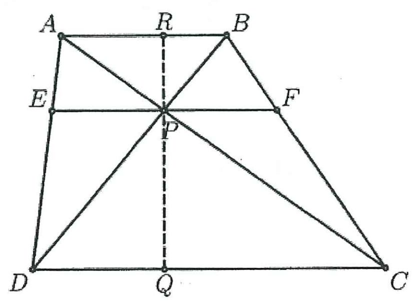
$$\frac{AE}{ED} = \frac{BF}{FC} = \frac{2}{3}.$$

Lines EF, AC, and BD all pass through a common point P. If segment DC has length 36, then what is the length of segment AB?



SOLUTION:

We note that triangles DPC and BPA are similar. We also note that line EF is parallel to AB and DC.



Let PQ be an altitude of triangle DPC, and let PR be an altitude of triangle APC. Since AB and DC are parallel, P, Q, and R all lie on the same line. Furthermore, the ratio of QP to PR is $3/2$, since EF, AB, and CD are parallel. Since triangles DPC and BPA are similar, it follows that the ratio of DC to AB is $3/2$. Since DC has length 36, it follows that the length of AB is 24. ■

5.

If a and b are integers such that $ab = 2008$, then what is the maximum possible value of $a + b$?

(A) 2009 (B) 2008 (C) 259 (D) 2007 (E) 1006

SOLUTION:

Without loss of generality, suppose that $a \leq b$. If $a = 1$, then $b = 2008$, and $a + b = 2009$. But if $a > 1$, then $a \geq 2$, so $a \leq b \leq 2008/2$, and $a+b \leq 2008 < 2009$. Evidently $a \neq 0$. If $a < 0$, then $b < 0$, and $a + b < 0 < 2009$. Since $a + b$ can equal 2009, and it must be less than 2009 when it does not equal 2009, our maximum is 2009. ■

6.

Suppose that segments DA, DP, and DS all have equal length. If angle PDS has measure $72°$, then what is the measure of angle PAS?

(A) $72°$ (B) $36°$ (C) $144°$ (D) $54°$ (E) Impossible to say

Recall:

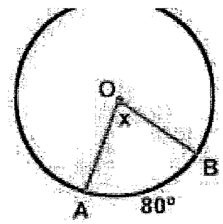
Central Angle

A central angle is an angle formed by two intersecting radii such that its vertex is at the center of the circle.

Central Angle = Intercepted Arc

$m\angle AOB = m\widehat{AD}$

$m\angle AOB = 80°$

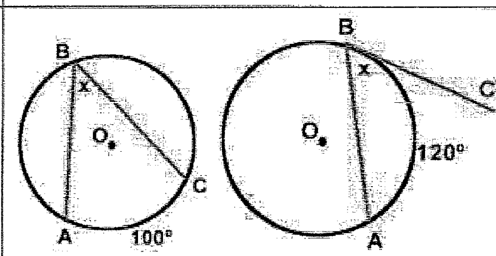


Inscribed Angle

An inscribed angle is an angle with its vertex "on" the circle, formed by two intersecting chords.

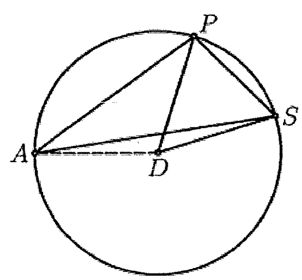
Inscribed Angle = $\frac{1}{2}$ Intercepted Arc

$m\angle ABC = \frac{1}{2} m\widehat{AC}$



Solution:

Let r be the length of DA, DP, and DS. Then by definition, A, P, and S lie on the circle with center D and radius r.



Since the measure of an inscribed angle in a circle is half the measure of the central angle that intercepts the same arc in, the measure of angle PAS is half the measure of angle PDS, so our answer is $72°/2 = 36°$. ■

7.

If x and y are positive integers such that

$$x^2 - 4y^2 = 36,$$

then what is the greatest possible value of $x + 2y$?

(A) 36 (B) 18 (C) 6 (D) 12 (E) 9

SOLUTION:

$$36 = x^2 - 4y^2 = (x - 2y)(x + 2y).$$

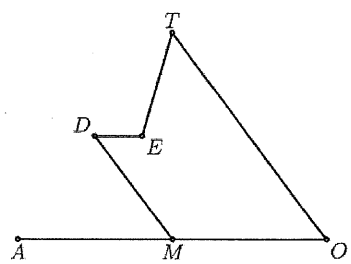
Now, since x and y are positive integers, $x+2y$ must also be positive, so $x-2y$ must be positive. Now, if $x-2y = 1$, then it is odd, so $x+2y$ must also be odd, a contradiction. So $x - 2y \geq 2$, and

$$x + 2y = \frac{36}{x - 2y} \leq \frac{36}{2} = 18.$$

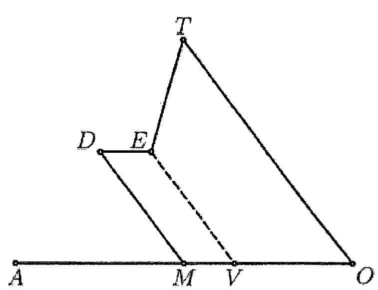
On the other hand, we can have $x + 2y = 18$ when $x = 10$ and $y = 4$. Therefore the greatest possible value of $x + y$ is 18. ■

8.

In the figure below, angles DMA, MDE, MOT, and OTE are all congruent. If segement MO has length 10 and segment ET has length 7, then what is the length of segment DE?



Let the line parallel to DM through point E intersect line AM at V.



Since angles DMA and MDE are congruent, lines DE and MV are parallel. Thus $DEMV$ is a parallelogram, so segments DE and MV have equal length.

Since angles DMA and MOT are congruent, lines DM and TO are parallel. Hence lines EV and TO are parallel, so $VOTE$ is a trapezoid. Since angles OTE and VOT are congruent, it is an isosceles trapezoid, so segment VO has the same length as segment ET, namely, 7. Since MO has length 10, it follow that segment MV must have length 3. Thus segment DE must also have length 3. ■

9.

Let $f(x) = \frac{2x-2}{x}$. What is $f^{2010}(2008)$?

Note : $f^n(x)$ means

$$\underbrace{f(f(\cdots f(x)\cdots))}_{n \text{ times}},$$

the composition of f with itself n times, evaluated at x. For example $f^3(x) = f(f(f(x)))$.

Solution

$$f(x) = \frac{2x-2}{x} = 2 - \frac{2}{x},$$

so

$$\begin{aligned} f^2(x) &= 2 - \frac{2}{2-\frac{2}{x}} \\ &= 2 - \frac{2x}{2x-2} \\ &= 2 - \frac{x}{x-1} = \frac{2x-2-x}{x-1} \\ &= \frac{x-2}{x-1} = 1 - \frac{1}{x-1}. \end{aligned}$$

Hence

$$\begin{aligned} f^4(x) = f^2(f^2(x)) &= 1 - \frac{1}{(1-\frac{1}{x-1})-1} \\ &= 1 - \frac{1}{-1/(x-1)} \\ &= 1 - (-x+1) \\ &= x. \end{aligned}$$

It then follows that for any integer n, $f^{4n}(x) = x$. Hence

$$f^{2010}(2008) = f^2(2008) = 1 - \frac{1}{2008-1} = \frac{2006}{2007}.$$