Luella 2015 Math Tournament Selected Problems 8/31/17 Thurs

2) The numbers 1, 3, 6, 10, 15, . . . are known as *triangular numbers*. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is:

A 494 B 495 C 496 D 497 E none of these

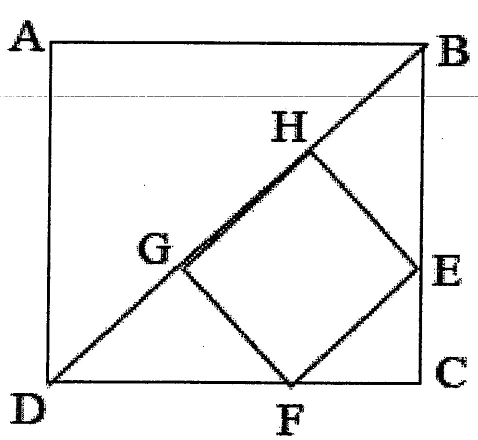
3) Solve $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9) = x$

A $\log_9 10$ B 2 C 9 D 4 E none of these

4) There is an angle θ, $0° < \theta < 90°$, such that $\tan\theta = \cos\theta$. What is the value of $\sin\theta$?

A $\frac{\sqrt{3}-1}{2}$ B $\frac{\sqrt{5}-1}{2}$ C $\frac{\sqrt{5}+1}{4}$ D $\frac{\sqrt{2}+1}{4}$ E it cannot be determined

6) If *ABCD* and *EFGH* are squares and *AB=1*, find the area of square *EFGH*.



A $\frac{\sqrt{3}}{2}$ B $\frac{3}{4}$ C $\frac{2\sqrt{2}}{5}$

D $\frac{3}{5}$ E $\frac{2}{9}$

7) The equation $x^3 - 3x + 1 = 0$ has three solutions: a, b, and c. Calculate $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

A 3 B $4\sqrt[3]{2}$ C $4\sqrt{3}$ D 8 E 9

9) Simplify $\sqrt{5 + \sqrt{5 + \sqrt{5 + \ldots}}}$

A $\frac{1+\sqrt{21}}{2}$ B $\frac{1+\sqrt{26}}{2}$ C 5 D ∞ E none of these

12) What is the maximum value of the function $f(x) = \frac{\sin^3 x \cos x}{\tan^2 x + 1}$?

A $\frac{1}{8}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D $\frac{1}{2}$ E 1

13) For how many integers m, with $10 \le m \le 100$, is $m^2 - m - 90$ divisible by 17?

A 7 B 8 C 9 D 10 E 11

16) Find the value of $\sin^2 10 + \sin^2 20 + \sin^2 30 + \ldots + \sin^2 80 + \sin^2 90$

A 1 B $\frac{\sqrt{3}}{2}$ C 3 D $\frac{\sqrt{2}}{2}$ E 5

19) Suppose $f(x)$ is a polynomial with integer coefficients for which 4 and 15 are both roots. Which of the following could possibly be the value of $f(12)$?

A 30 B 72 C 12 D 36 E none of these

25) How many integers x in $\{1, 2, 3, \ldots, 99, 100\}$ are there such that $x^2 + x^3$ is the square of an integer?

A 7 B 8 C 9 D 10 E 11

27) Given that $f(x) = ax + b$ with a and b real, if $f(f(f(x))) = 27x + 52$, find the sum $a + b$.

A 3 B 4 C 5 D 6 E 7

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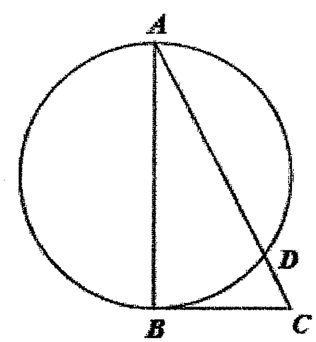
28) Suppose $\overline{AB}$ is a diameter of the circle shown, $\overline{BC}$ is tangent to the circle, $\angle BAC = 30°$, and $CD = \sqrt{3}$. What is the distance from A to B?



A $3\sqrt{3}$ B 6 C $4\sqrt{3}$ D 8 E $5\sqrt{3}$

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29) Given that a is a nonzero real number such that $\sin x + \sin y = a$ and $\cos x + \cos y = 2a$, find the value of $\cos(x - y)$.

A $\frac{5a^2-2}{2}$ B $\frac{a^2-2}{2}$ C $\frac{3a^2-2}{2}$ D $\frac{9a^2-2}{2}$ E $\frac{7a^2-2}{2}$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Ciphering #4

The number $\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{\ddots}}}}}$ can be expressed in the form $\frac{a+\sqrt{b}}{c}$, where a, b and c are integers

Find $a + b + c$ (Note, the pattern goes on forever.)

Ciphering #8

Simplify

$$\frac{1}{\sqrt{2}+\sqrt{1}}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{36}+\sqrt{35}}$$

Ciphering #15

If $\sin(2a) = \frac{1}{7}$, compute the numerical value of

$sin^4(a) + cos^4(a)$.

Luella 2015 Math Tournament Selected Problems 8/31/17 Thurs Key

2) The numbers 1, 3, 6, 10, 15, . . . are known as *triangular numbers*. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is:

A 494 B 495 $\boxed{\text{C } 496}$ D 497 E none of these

$\frac{n(n+1)}{2} = 500$ $n(n+1) \leq 1000$ $n \approx 30$ guess and check $\rightarrow$ $\frac{31 \times 32}{2} = \frac{992}{2} = 496$

3) Solve $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9) = x$

A $\log_9 10$ $\boxed{\text{B } 2}$ C 9 D 4 E none of these

change of base: $\log_b a = \frac{\log a}{\log b}$ $\frac{\log 4}{\log 3} \cdot \frac{\log 5}{\log 4} \cdot \ldots\ldots \frac{\log 8}{\log 7} \cdot \frac{\log 9}{\log 8} = \frac{\log 9}{\log 3} = \log_3 9 = \log_3 3^2 = \boxed{2}$

4) There is an angle θ, $0° < \theta < 90°$, such that $\tan\theta = \cos\theta$. What is the value of $\sin\theta$?

A $\frac{\sqrt{3}-1}{2}$ $\boxed{\text{B } \frac{\sqrt{5}-1}{2}}$ C $\frac{\sqrt{5}+1}{4}$ D $\frac{\sqrt{2}+1}{4}$ E it cannot be determined

$\tan\theta = \cos\theta$

$\frac{\sin\theta}{\cos\theta} = \cos\theta$

$\sin\theta = \cos^2\theta$

$\sin\theta = 1 - \sin^2\theta$

$\sin^2\theta + \sin\theta - 1 = 0$

let $x = \sin\theta$

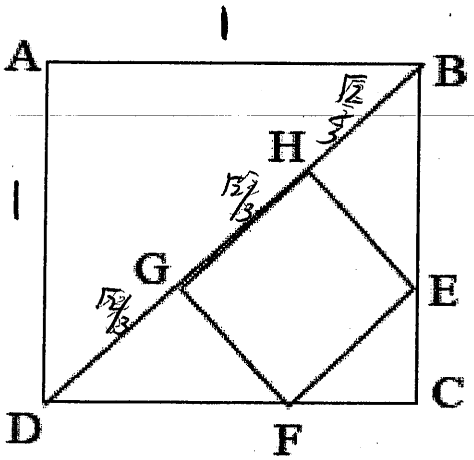
$x^2 + x - 1 = 0$

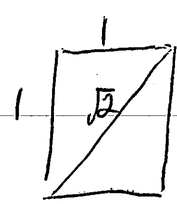
$\frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$

$\frac{-1 \pm \sqrt{5}}{2} = x$

$\boxed{\sin\theta = \frac{\sqrt{5}-1}{2}}$

6) If *ABCD* and *EFGH* are squares and *AB=1*, find the area of square *EFGH*.





A $\frac{\sqrt{3}}{2}$ B $\frac{3}{4}$ C $\frac{2\sqrt{2}}{5}$

$\left(\frac{\sqrt{2}}{3}\right)^2 = \frac{2}{9}$ D $\frac{3}{5}$ $\boxed{\text{E } \frac{2}{9}}$

7) The equation $x^3-3x+1=0$ has three solutions: a, b, and c. Calculate $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

$(\frac{-b}{a})$ sum of roots $\Rightarrow a+b+c=\frac{0}{1}=0$ product of roots $(abc)=\frac{-1}{1}=-1$ $ab+bc+bc=\frac{-3}{1}=-3$

A 3 B $4\sqrt[3]{2}$ C $4\sqrt{3}$ D 8 $\boxed{\text{E } 9}$

$$\frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}=\frac{(ab+bc+ac)^2-2abc(a+b+c)}{(abc)^2}=\frac{(-3)^2-2(-1)(0)}{(-1)^2}=\frac{9+0}{1}=\boxed{9}$$

$*(ab+bc+ac)^2=a^2b^2+b^2c^2+a^2c^2+2abc(a+b+c)$

9) Simplify $\sqrt{5+\sqrt{5+\sqrt{5+\dots}}}$

$x=\sqrt{5+x}$

$x^2=5+x$

$x^2-x-5=0$

$\frac{1\pm\sqrt{1^2-4(1)(-5)}}{2(1)}=\frac{1\pm\sqrt{21}}{2}$

A $\frac{1+\sqrt{21}}{2}$ B $\frac{1+\sqrt{26}}{2}$ C 5 D ∞ E none of these

$x=\sqrt{5+x}$

$x^2=5+x$

$x^2-x-5=0$

$\frac{1\pm\sqrt{1^2-4(1)(-5)}}{2(1)}=\boxed{\frac{1\pm\sqrt{21}}{2}}$

#2) $\frac{\sin^3x\cos x}{\sec^2x}$

$=\sin^3x\cos x\cdot\cos^2x$

$=\sin^3x\cos^3x$

$=\left(\frac{\sqrt2}{2}\right)^3\left(\frac{\sqrt2}{2}\right)^3$

$=\left(\frac{\sqrt2}{2}\right)^6=\left(\frac{1}{\sqrt2}\right)^6=\boxed{\frac{1}{8}}$

$1+\tan^2x=\sec^2x$

12) What is the maximum value of the function $f(x)=\frac{\sin^3x\cos x}{\tan^2x+1}$?

$\boxed{\text{A } \frac{1}{8}}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D $\frac{1}{2}$ E 1

m^2-m-90

$(m+9)(m-10)$

$m+9=0$ $m-10=0$

$m+9=17$

$m=\cancel{8}, 25, 42$

$59, 76, 93$

$m-10=17$

$27, 44, 61, 78, 95$ create multiples of 17

13) For how many integers m, with $\boxed{10\le m\le 100}$, is m^2-m-90 divisible by 17?

A 7 B 8 C 9 $\boxed{\text{D } 10}$ $\boxed{\text{E } 11}$

multiples of 17

$(m+9)$ and $(m-10)$ is divisible by 17

	0	17	34	51	68	85	102
m+9	~~-9~~	~~8~~	25	42	59	76	93
m-10	10	27	44	61	78	95	~~112~~

$\sin x=\cos(90-x)$

16) Find the value of $\sin^2 10+\sin^2 20+\sin^2 30+\dots+\sin^2 80+\sin^2 90$

$\sin^2x+\cos^2x=1$

$\sin^2x+\sin^2(90-x)=1$

$*\sin^2 10+\sin^2(90-10)=1$

$\sin^2 10+\sin^2 80=1$

A 1 B $\frac{\sqrt3}{2}$ C 3 D $\frac{\sqrt2}{2}$ E 5

$\sin^2 10$ $\sin^2 20$ $\sin^2 30$ $\sin^2 40$ $\sin^2 90$

$+\sin^2 80$ $+\sin^2 70$ $+\sin^2 60$ $+\sin^2 50$

$1+1+1+1+1=\boxed{5}$

19) Suppose $f(x)$ is a polynomial with integer coefficients for which 4 and 15 are both roots. Which of the following could possibly be the value of $f(12)$?

integer coefficient

A 30 $\boxed{\text{B } 72}$ C 12 D 36 E none of these

$f(x) = k(x-4)(x-15)$

$f(12) = k(12-4)(12-15)$

$f(12) = k(8)(-3)$

$f(12) = -24k$

$72 = -24k$

$-3 = k$

25) How many integers x in $\{1, 2, 3, \ldots, 99, 100\}$ are there such that $x^2 + x^3$ is the square of an integer?

A 7 B 8 $\boxed{\text{C } 9}$ D 10 E 11

$x^2 + x^3$

$= x^2(1+x)$

* Count how many x's where $x+1$ is a perfect square

$x = 3, 8, 15, 24, 35, 48, 63, 80, 99 \rightarrow \boxed{9 \text{ numbers}}$

$2^2=4$, $3^2=9$, $4^2=16$, $5^2=25$, $6^2=36$, $7^2=49$, $8^2=64$, $9^2=81$, $10^2=100$

27) Given that $f(x) = ax + b$ with a and b real, if $f(f(f(x))) = 27x + 52$, find the sum $a + b$.

A 3 $\boxed{\text{B } 4}$ C 5 D 6 E 7

$a[a[ax+b]+b]+b$

$a^3x + a^2b + ab + b = 27x + 52$

$a^3 = 27$

$a = 3$

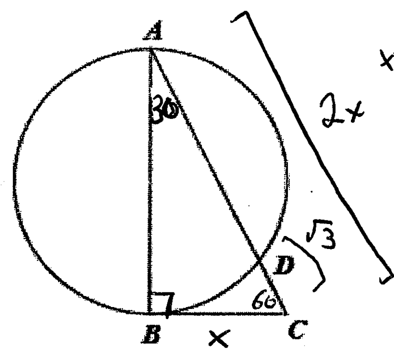
$a^2b + ab + b = 52$

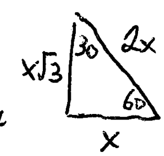
$3^2b + 3b + b = 52$

$13b = 52$

$\boxed{b = 4}$

28) Suppose $\overline{AB}$ is a diameter of the circle shown, $\overline{BC}$ is tangent to the circle, $\angle BAC = 30°$, and $CD = \sqrt{3}$. What is the distance from A to B?





A $3\sqrt{3}$ $\boxed{\text{B } 6}$ C $4\sqrt{3}$ D 8 E $5\sqrt{3}$

Secant-tangent to circle

$BC \cdot BC = CD \cdot AC$

$x \cdot x = (\sqrt{3})(2x)$

$x^2 = 2x\sqrt{3}$

$x^2 - 2x\sqrt{3} = 0$

$x(x - 2\sqrt{3}) = 0$

$x = 2\sqrt{3}$

$AB = \text{long leg} = x\sqrt{3} = (2\sqrt{3})(\sqrt{3}) = 2 \cdot 3 = \boxed{6}$

29) Given that a is a nonzero real number such that $\sin x + \sin y = a$ and $\cos x + \cos y = 2a$, find the value of $\cos(x-y)$. * $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\boxed{\text{A } \frac{5a^2-2}{2}}$ B $\frac{a^2-2}{2}$ C $\frac{3a^2-2}{2}$ D $\frac{9a^2-2}{2}$ E $\frac{7a^2-2}{2}$

$(\sin x + \sin y)^2 = a^2$

add together

$\sin^2 x + 2\sin x \sin y + \sin^2 y = a^2$

$(\cos x + \cos y)^2 = (2a)^2$

$\cos^2 x + 2\cos x \cos y + \cos^2 y = 4a^2$

$\underline{\sin^2 x + \cos^2 x} + 2\sin x \sin y + 2\cos x \cos y + \underline{\sin^2 y + \cos^2 y} = 5a^2$

$1 + 2[\cos(x-y)] + 1 = 5a^2$

$2\cos(x-y) + 2 = 5a^2$

$2(\cos(x-y) + 1) = 5a^2$

$\cos(x-y) + 1 = \frac{5}{2}a^2$

$\cos(x-y) = \frac{5a^2}{2} - \frac{2}{2} = \boxed{\frac{5a^2-2}{2}}$

Ciphering #4

The number $\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{4+\dfrac{3}{\ddots}}}}}$ can be expressed in the form $\frac{a+\sqrt{b}}{c}$, where a, b and c are integers

Find $a + b + c$ (Note, the pattern goes on forever.)

$x = \dfrac{3}{4+x}$

$4x + x^2 = 3$

$x^2 + 4x - 3 = 0$

$\dfrac{-4 \pm \sqrt{16-4(1)(-3)}}{2(1)}$

$\dfrac{-4 \pm \sqrt{28}}{2}$

$a=-4,\ b=28,\ c=2$

$a+b+c =$

$-4+28+2 = \boxed{26}$

Ciphering #8

Simplify

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{36}+\sqrt{35}}$$

$\dfrac{1}{\sqrt{2}+\sqrt{1}} \cdot \dfrac{\sqrt{2}-\sqrt{1}}{\sqrt{2}-\sqrt{1}} = \dfrac{\sqrt{2}-\sqrt{1}}{1}$

$\dfrac{\cancel{\sqrt{2}}-\sqrt{1}+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\cancel{\sqrt{4}}-\cancel{\sqrt{3}}+\ldots+\cancel{\sqrt{35}}-\cancel{\sqrt{34}}+\sqrt{36}-\cancel{\sqrt{35}}}{1}$

$\dfrac{1-\sqrt{36}}{1} = 1-6 = \boxed{-5}$

Ciphering #15

If $\sin(2a) = \frac{1}{7}$ compute the numerical value of

$\sin^4(a) + \cos^4(a)$.

$\sin 2a = 2\sin a\cos a = \frac{1}{7}$

$\sin 2a = (2\sin a\cos a)^2 = \left(\frac{1}{7}\right)^2$

$4\sin^2 a\cos^2 a = \frac{1}{49}$

$2\sin^2 a\cos^2 a = \frac{1}{49 \cdot 2}$

$2\sin^2 a\cos^2 a = \frac{1}{98}$

$\sin^2 a + \cos^2 a = 1$

$(\sin^2 a + \cos^2 a)^2 = (1)^2$

$\sin^4 a + 2\sin^2 a\cos^2 a + \cos^4 a = 1$

$\sin^4 a + \cos^4 a = 1 - 2\sin^2 a\cos^2 a$

$= 1 - \frac{1}{98}$

$= \boxed{\frac{97}{98}}$