Calculus Notes Ch. 2.6 - Related Rates

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

1. Write what you are given
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time t
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

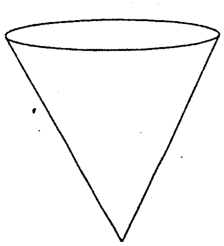
Example 1: The sides of a square are increasing at a rate of 5 cm/min. How fast is the area increasing when the sides measure 15 cm in length?

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.



Solution Key

Calculus Notes Ch. 2.6 - Related Rates

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

*use units of measurement to help match appropriate variable to values

1. Write what you are given
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time *t*
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

Example 1: The sides of a square are increasing at a rate of 5 cm/min. How fast is the area increasing when the sides measure 15 cm in length?

$A = x^2$

represents change in Area w/ respect to time

represents change in side length with respect to time

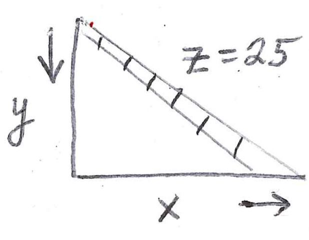
$\frac{dA}{dt} = 2x\left(\frac{dx}{dt}\right)$

Given: $\frac{dx}{dt} = 5\ \text{cm/min}$ Find $\frac{dA}{dt} = \_\_\_$

$x = 15\text{cm}$

$\frac{dA}{dt} = 2(15)(5) = \boxed{150\text{cm}^2/\text{min}}$

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?



$x = 15$

$y = 20$

$z = 25$

$15^2 + y^2 = 25^2$

$y = 20$

$\frac{dx}{dt} = 3$

$\frac{dy}{dt} = \_\_\_$

$\frac{dz}{dt} = 0$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$2(15)(3) + 2(20)\frac{dy}{dt} = 2(25)(0)$

$40\left(\frac{dy}{dt}\right) = -90$ $\boxed{\frac{dy}{dt} = -9/4\ \text{ft/s}}$

$A = \frac{1}{2}xy$ $f'g + fg'$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)(y) + \frac{1}{2}(x)\left(\frac{dy}{dt}\right)$

$\frac{dA}{dt} = \frac{1}{2}(3)(20) + \frac{1}{2}(15)(-9/4)$

$= 30 - \frac{135}{8} = \boxed{\frac{105}{8}\ \text{ft}^2/\text{s}}$

$\approx 13.125\ \text{ft}^2/\text{s}$

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

change in volume (cm^3/s)

$V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\frac{dV}{dt} = 10\, cm^3/s$

$\frac{dS}{dt} = \_\_\_$

$S = 64\pi$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$

$64\pi = 4\pi r^2$

$16 = r^2$

$r = 4$

$10 = 4\pi(4)^2\left(\frac{dr}{dt}\right)$

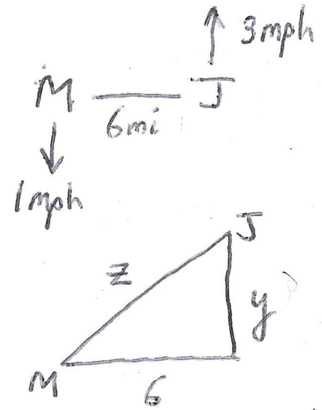
$\frac{10}{64\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{32\pi}$

$\frac{dS}{dt} = 8\pi(4)\left(\frac{5}{32\pi}\right)$

$\boxed{\frac{dS}{dt} = 5\, cm^2/sec}$

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?



$\frac{dy}{dt} = 4$ mph

$y = (4)(2) = 8$ mi

$z = 10$ mi

$6^2 + 8^2 = z^2$

$z = 10$

Find $\frac{dz}{dt} = \_\_\_$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

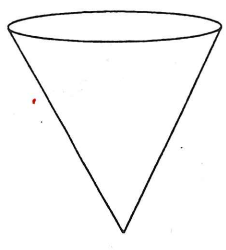
$\cancel{2(6)(0)} + 2(8)(4) = 2(10)\left(\frac{dz}{dt}\right)$

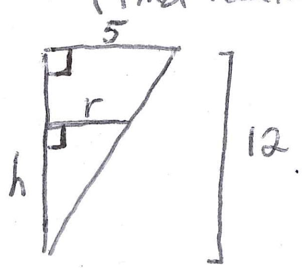
$64 = 20\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{64}{20} = \frac{16}{5} = \boxed{3.2 \text{ mph}}$

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

*use similiar triangles to rewrite equation using less variables.
(find relationship between height and radius, use substitution)





$V = \frac{\pi}{3}r^2 h$

$\frac{dV}{dt} = 10\, ft^3/min$

$\frac{dh}{dt} = \_\_\_$

$h = 8$ ft.

$\frac{r}{5} = \frac{h}{12}$

$5h = 12r$

$\frac{5}{12}h = r$

*Rewrite volume in terms of "h"

$V = \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h = \frac{\pi}{3} \cdot \frac{25h^2}{144} \cdot h = \frac{25\pi}{432}h^3$

$V = \frac{25\pi}{432}h^3$

$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2\left(\frac{dh}{dt}\right)$

$10 = \frac{25\pi}{432} \cdot 3(8)^2 \frac{dh}{dt}$

$10 = \frac{4800\pi}{432} \cdot \frac{dh}{dt}$

$10 \cdot \frac{432}{4800\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{9}{10\pi}\, ft/min}$