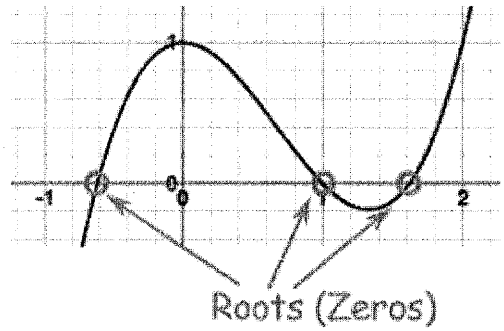
Polynomials: Sums and Products of Roots



A "root" (or "zero") is where the polynomial **is equal to zero**:

For $f(x) = a(x-p)(x-q)(x-r)$, the roots(or solutions) are p, q, and r

Given quadratic equation (polynomial of degree of 2): $\mathbf{ax^2 + bx + c = 0}$

$a(x-p)(x-q) = a(x^2 - qx - px + pq) = a(x^2 - (p+q)x + pq)$

Adding the roots gives: $-\frac{b}{a}$ **Multiplying the roots gives:** $\frac{c}{a}$

Example 1: $f(x) = 2x^2 + 5x - 3$ Find the roots, and confirm properties above:

Now let us look at a Cubic (one degree higher than Quadratic): $\mathbf{ax^3 + bx^2 + cx + d}$

$a(x-p)(x-q)(x-r) = a(x-p)(x^2 - qx - qr + qr) = a(x^3 - qx^2 - qrx + qrx - px^2 + pqx + pqr - pqr)$

$ax^3 - a(p+q+r)x^2 + a(pq+pr+qr)x - a(pqr)$

And we get:

Cubic:	$\mathbf{ax^3}$	$\mathbf{+ bx^2}$	$\mathbf{+cx}$	$\mathbf{+d}$
Expanded Factors:	$\mathbf{ax^3}$	$\mathbf{-a(p+q+r)x^2}$	$\mathbf{+a(pq+pr+qr)x}$	$\mathbf{-a(pqr)}$

Properties of Cubic Roots

a) Adding the roots gives: $-\frac{b}{a}$ (same as quadratic) $\mathbf{p + q + r} = -\frac{b}{a}$

b) Multiplying the roots gives: $-\frac{d}{a}$ (similar to $\frac{+c}{a}$ of quadratic) $\mathbf{pqr} = -\frac{d}{a}$

c) We also get $\mathbf{pq + pr + qr} = \frac{+c}{a}$ (another useful property)

Cubic:	ax^3	$+ bx^2$	$+cx$	$+d$
Expanded Factors:	ax^3	$-a(p+q+r)x^2$	$+a(pq+pr+qr)x$	$-a(pqr)$

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Example 2: Luella 2015 Varsity Problem #7

7) The equation $x^3 - 3x + 1 = 0$ has three solutions: a, b, and c. Calculate $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

A 3 B $4\sqrt[3]{2}$ C $4\sqrt{3}$ D 8 E 9

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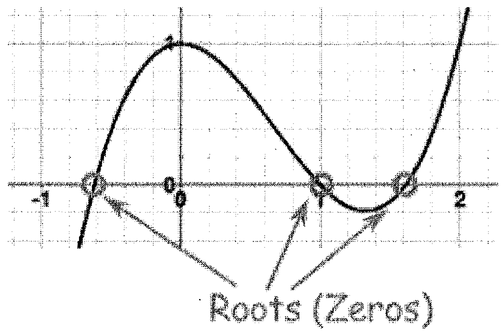
Key



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Example 1: $f(x) = 2x^2 + 5x - 3$ Find the roots, and confirm properties above:

$(2x-1)(x+3)$

$x = \frac{1}{2}, -3$

product: $\frac{1}{2} \times -3 = -\frac{3}{2} = \frac{c}{a} = \frac{-3}{2}$ ✓

sum $= -3 + \frac{1}{2} = -\frac{5}{2} = \frac{-b}{a} = -\frac{(5)}{2} = -\frac{5}{2}$

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$1x^3 + 0x^2 - 3x + 1 = 0$ $a=1$ $b=0$ $c=-3$ $d=1$

sum of roots $= \frac{-b}{a} = \frac{0}{1} = 0$

product of roots $= \frac{-d}{a} = \frac{-1}{1} = -1$

$ab + bc + ac = \frac{c}{a} = \frac{-3}{1} = -3$

$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2} =$

$*(ab+bc+ac)^2 = (ab+bc+ac)(ab+bc+ac)$

$= a^2b^2 + b^2c^2 + a^2c^2 + 2abc(a+b+c)$

$a^2b^2 + b^2c^2 + a^2c^2 = (ab+bc+ac)^2 - 2abc(a+b+c)$

$\rightarrow \frac{(ab+bc+ac)^2 - 2abc(a+b+c)}{(abc)^2}$

$= \frac{(-3)^2 - 2(-1)(0)}{(-1)^2} = \frac{9}{1} = \boxed{9}$