1)

Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function f.

(a) $f'(x) = 0$ at $x = 4$

$f'(x) > 0$ for $0 < x < 4$

$f'(x) < 0$ for $x > 4$

Therefore f has a relative maximum at $x = 4$.

$$3 : \begin{cases} 1 : x = 4 \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$$

(b) $$f''(x) = -x^{-3} + (4 - x)\left(-3x^{-4}\right)$$
$$= -x^{-3} - 12x^{-4} + 3x^{-3}$$
$$= 2x^{-4}(x - 6)$$
$$= \frac{2(x-6)}{x^4}$$

$f''(x) < 0$ for $0 < x < 6$

The graph of f is concave down on the interval $0 < x < 6$.

$$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with justification} \end{cases}$$

(c) $$f(x) = 2 + \int_1^x \left(4t^{-3} - t^{-2}\right) dt$$
$$= 2 + \left[-2t^{-2} + t^{-1}\right]_{t=1}^{t=x}$$
$$= 3 - 2x^{-2} + x^{-1}$$

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

2)

The function g is defined for $x > 0$ with $g(1) = 2$, $g'(x) = \sin\left(x + \frac{1}{x}\right)$, and $g''(x) = \left(1 - \frac{1}{x^2}\right)\cos\left(x + \frac{1}{x}\right)$.

(a) Find all values of x in the interval $0.12 \le x \le 1$ at which the graph of g has a horizontal tangent line.

(b) On what subintervals of $(0.12, 1)$, if any, is the graph of g concave down? Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 0.3$.

(d) Does the line tangent to the graph of g at $x = 0.3$ lie above or below the graph of g for $0.3 < x < 1$? Why?

(a) The graph of g has a horizontal tangent line when $g'(x) = 0$. This occurs at $x = 0.163$ and $x = 0.359$.	$2 : \begin{cases} 1 : \text{sets } g'(x) = 0 \\ 1 : \text{answer} \end{cases}$
(b) $g''(x) = 0$ at $x = 0.129458$ and $x = 0.222734$ The graph of g is concave down on $(0.1295, 0.2227)$ because $g''(x) < 0$ on this interval.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$
(c) $g'(0.3) = -0.472161$ $g(0.3) = 2 + \int_1^{0.3} g'(x)\,dx = 1.546007$ An equation for the line tangent to the graph of g is $y = 1.546 - 0.472(x - 0.3)$.	$4 : \begin{cases} 1 : g'(0.3) \\ 1 : \text{integral expression} \\ 1 : g(0.3) \\ 1 : \text{equation} \end{cases}$
(d) $g''(x) > 0$ for $0.3 < x < 1$ Therefore the line tangent to the graph of g at $x = 0.3$ lies below the graph of g for $0.3 < x < 1$.	1 : answer with reason

3)

Let f be the function given by $f(x) = \frac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by

$f'(x) = \frac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim_{x \to 0^+} f(x)$.

(a) $f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$, $f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = -\frac{1}{e^4}$ An equation for the tangent line is $y = \frac{2}{e^2} - \frac{1}{e^4}(x - e^2)$.	$2 : \begin{cases} 1 : f(e^2) \text{ and } f'(e^2) \\ 1 : \text{answer} \end{cases}$
(b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.	$3 : \begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$
(c) $f''(x) = \frac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \frac{-3 + 2\ln x}{x^3}$ for all $x > 0$ $f''(x) = 0$ when $-3 + 2\ln x = 0$ $x = e^{3/2}$ The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.	$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$
(d) $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ or Does Not Exist	1 : answer

4)

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

(a) $f'(x) = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$ $f''(x) = -\dfrac{1}{4}kx^{-3/2} + x^{-2}$	$2: \begin{cases} 1: f'(x) \\ 1: f''(x) \end{cases}$
(b) $f'(1) = \dfrac{1}{2}k - 1 = 0 \Rightarrow k = 2$ When $k = 2$, $f'(1) = 0$ and $f''(1) = -\dfrac{1}{2} + 1 > 0$. f has a relative minimum value at $x = 1$ by the Second Derivative Test.	$4: \begin{cases} 1: \text{sets } f'(1) = 0 \text{ or } f'(x) = 0 \\ 1: \text{solves for } k \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(c) At this inflection point, $f''(x) = 0$ and $f(x) = 0$. $f''(x) = 0 \Rightarrow \dfrac{-k}{4x^{3/2}} + \dfrac{1}{x^2} = 0 \Rightarrow k = \dfrac{4}{\sqrt{x}}$ $f(x) = 0 \Rightarrow k\sqrt{x} - \ln x = 0 \Rightarrow k = \dfrac{\ln x}{\sqrt{x}}$ Therefore, $\dfrac{4}{\sqrt{x}} = \dfrac{\ln x}{\sqrt{x}}$ $\Rightarrow 4 = \ln x$ $\Rightarrow x = e^4$ $\Rightarrow k = \dfrac{4}{e^2}$	$3: \begin{cases} 1: f''(x) = 0 \text{ or } f(x) = 0 \\ 1: \text{equation in one variable} \\ 1: \text{answer} \end{cases}$