1)

(a) $f'(x) = -2\sin(2x) + \cos x\, e^{\sin x}$

$f'(\pi) = -2\sin(2\pi) + \cos \pi\, e^{\sin \pi} = -1$

$2 : f'(\pi)$

(b) $k'(x) = h'(f(x)) \cdot f'(x)$

$k'(\pi) = h'(f(\pi)) \cdot f'(\pi) = h'(2) \cdot (-1)$

$= \left(-\frac{1}{3}\right)(-1) = \frac{1}{3}$

$2 : \begin{cases} 1 : k'(x) \\ 1 : k'(\pi) \end{cases}$

(c) $m'(x) = -2g'(-2x) \cdot h(x) + g(-2x) \cdot h'(x)$

$m'(2) = -2g'(-4) \cdot h(2) + g(-4) \cdot h'(2)$

$= -2(-1)\left(-\frac{2}{3}\right) + 5\left(-\frac{1}{3}\right) = -3$

$3 : \begin{cases} 2 : m'(x) \\ 1 : m'(2) \end{cases}$

(d) g is differentiable. $\Rightarrow$ g is continuous on the interval $[-5, -3]$.

$\frac{g(-3) - g(-5)}{-3 - (-5)} = \frac{2 - 10}{2} = -4$

Therefore, by the Mean Value Theorem, there is at least one value c, $-5 < c < -3$, such that $g'(c) = -4$.

$2 : \begin{cases} 1 : \frac{g(-3) - g(-5)}{-3 - (-5)} \\ 1 : \text{justification, using Mean Value Theorem} \end{cases}$

2)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-6	3	2	8
2	2	-2	-3	0
3	8	7	6	2
6	4	5	3	-1

The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

(a) $k(3) = f(g(3)) = f(6) = 4$

$k'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot 2 = 5 \cdot 2 = 10$

An equation for the tangent line is $y = 10(x - 3) + 4$.

$3 : \begin{cases} 2 : \text{slope at } x = 3 \\ 1 : \text{equation for tangent line} \end{cases}$

(b) $$h'(1) = \frac{f(1) \cdot g'(1) - g(1) \cdot f'(1)}{(f(1))^2} = \frac{(-6) \cdot 8 - 2 \cdot 3}{(-6)^2} = \frac{-54}{36} = -\frac{3}{2}$$

$3 : \begin{cases} 2 : \text{expression for } h'(1) \\ 1 : \text{answer} \end{cases}$

(c) $$\int_1^3 f''(2x)\,dx = \frac{1}{2}\left[f'(2x)\right]_1^3 = \frac{1}{2}[f'(6) - f'(2)] = \frac{1}{2}[5 - (-2)] = \frac{7}{2}$$

$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

3)

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.

(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.

(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.

(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

(a) $x = 1$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a relative minimum at $x = 1$.

1 : answer with justification

(b) f' is differentiable $\Rightarrow$ f' is continuous on the interval $-1 \le x \le 1$

$$\frac{f'(1) - f'(-1)}{1 - (-1)} = \frac{0 - 0}{2} = 0$$

Therefore, by the Mean Value Theorem, there is at least one value c, $-1 < c < 1$, such that $f''(c) = 0$.

$2 : \begin{cases} 1 : f'(1) - f'(-1) = 0 \\ 1 : \text{explanation, using Mean Value Theorem} \end{cases}$

(c) $h'(x) = \dfrac{1}{f(x)} \cdot f'(x)$

$$h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d)
$$\int_{-2}^{3} f'(g(x))g'(x)\,dx = \Big[f(g(x))\Big]_{x=-2}^{x=3}$$
$$= f(g(3)) - f(g(-2))$$
$$= f(1) - f(-1)$$
$$= 2 - 8 = -6$$

$3 : \begin{cases} 2 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

4)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	– 4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2 : \begin{cases} 1 : \dfrac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.

$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3 : \begin{cases} 1 : g^{-1}(2) \\ 1 : \left(g^{-1}\right)'(2) \\ 1 : \text{tangent line equation} \end{cases}$

5)

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2h(x) + 3x^3h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

3 : { 1 : form of product rule; 1 : $a'(x)$; 1 : $a'(2)$ }

(c) Because h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$.

Also, $\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim_{x\to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim_{x\to 2} \left(1 - (f(x))^3\right) = 0$.

Thus $\lim_{x\to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim_{x\to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus $f'(2) = -\frac{1}{3}$.

4 : { 1 : $\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$; 1 : $f(2)$; 1 : L'Hospital's Rule; 1 : $f'(2)$ }

(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x\to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

1 : continuous with justification