

1)

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume  $V$  of a right circular cylinder with radius  $r$  and height  $h$  is given by  $V = \pi r^2 h$ .)

- (a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?
- (b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is  $R(t) = 400\sqrt{t}$  cubic centimeters per minute, where  $t$  is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time  $t$  when the oil slick reaches its maximum volume. Justify your answer.
- (c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

- (a) When  $r = 100$  cm and  $h = 0.5$  cm,  $\frac{dV}{dt} = 2000$  cm<sup>3</sup>/min  
and  $\frac{dr}{dt} = 2.5$  cm/min.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$

$$4 : \begin{cases} 1 : \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2 : \text{expression for } \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$$

- (b)  $\frac{dV}{dt} = 2000 - R(t)$ , so  $\frac{dV}{dt} = 0$  when  $R(t) = 2000$ .

This occurs when  $t = 25$  minutes.

Since  $\frac{dV}{dt} > 0$  for  $0 < t < 25$  and  $\frac{dV}{dt} < 0$  for  $t > 25$ ,

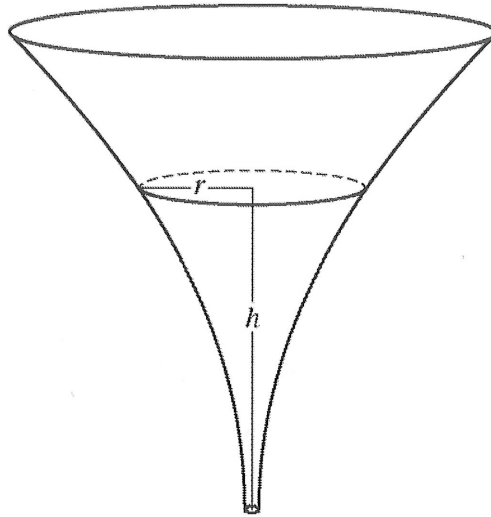
the oil slick reaches its maximum volume 25 minutes after the device begins working.

$$3 : \begin{cases} 1 : R(t) = 2000 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

- (c) The volume of oil, in cm<sup>3</sup>, in the slick at time  $t = 25$  minutes is given by  $60,000 + \int_0^{25} (2000 - R(t)) dt$ .

$$2 : \begin{cases} 1 : \text{limits and initial condition} \\ 1 : \text{integrand} \end{cases}$$

2)



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height  $h$ , the radius of the funnel is given by  $r = \frac{1}{20}(3 + h^2)$ , where  $0 \leq h \leq 10$ . The units of  $r$  and  $h$  are inches.

- (a) Find the average value of the radius of the funnel.  
 (b) Find the volume of the funnel.  
 (c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is  $h = 3$  inches, the radius of the surface of the liquid is decreasing at a rate of  $\frac{1}{5}$  inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\begin{aligned} \text{(a) Average radius} &= \frac{1}{10} \int_0^{10} \frac{1}{20}(3 + h^2) dh = \frac{1}{200} \left[ 3h + \frac{h^3}{3} \right]_0^{10} \\ &= \frac{1}{200} \left( \left( 30 + \frac{1000}{3} \right) - 0 \right) = \frac{109}{60} \text{ in} \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(b) Volume} &= \pi \int_0^{10} \left( \left( \frac{1}{20} \right) (3 + h^2) \right)^2 dh = \frac{\pi}{400} \int_0^{10} (9 + 6h^2 + h^4) dh \\ &= \frac{\pi}{400} \left[ 9h + 2h^3 + \frac{h^5}{5} \right]_0^{10} \\ &= \frac{\pi}{400} \left( \left( 90 + 2000 + \frac{100000}{5} \right) - 0 \right) = \frac{2209\pi}{40} \text{ in}^3 \end{aligned}$$

3 :  $\begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \text{(c) } \frac{dr}{dt} &= \frac{1}{20}(2h) \frac{dh}{dt} \\ -\frac{1}{5} &= \frac{3}{10} \frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3} \text{ in/sec} \end{aligned}$$

3 :  $\begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

3)

4pts.

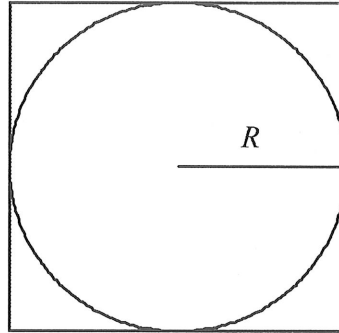
(a)  $P = 8R$

$$\frac{dP}{dt} = 8 \frac{dR}{dt}$$

$$6 = \frac{dC}{dt} = 2\pi \frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{3}{\pi}; \quad \frac{dP}{dt} = \frac{24}{\pi} \text{ inches/second}$$

$$\approx 7.639 \text{ inches/second}$$



5pts

(b)  $\text{Area} = 4R^2 - \pi R^2$

$$\frac{d(\text{Area})}{dt} = 8R \frac{dR}{dt} - 2\pi R \frac{dR}{dt}$$

$$= (4 - \pi) 2R \frac{dR}{dt}$$

$$\text{Area of circle} = 25\pi = \pi R^2$$

$$R = 5$$

$$\frac{d(\text{Area})}{dt} = \frac{120}{\pi} - 30 \text{ inches}^2/\text{second}$$

$$= (4 - \pi) \frac{30}{\pi} \text{ inches}^2/\text{second}$$

$$\approx 8.197 \text{ inches}^2/\text{second}$$

4) Solution

3 pts

$$(a) \frac{r}{h} = \frac{4}{12} = \frac{1}{3} \quad r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{\pi h^3}{27}$$

3 pts

$$(b) \frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}$$

$$= \frac{\pi h^2}{9} (h-12) = -9\pi$$

$V$  is decreasing at  $9\pi \text{ ft}^3 / \text{min}$

3 pts

(c) Let  $W$  = volume of water in cylindrical tank

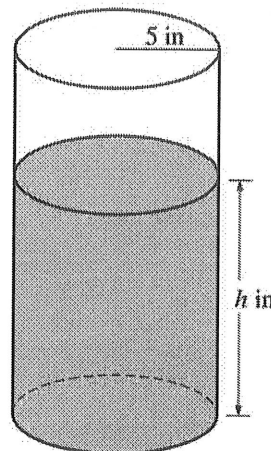
$$W = 400\pi y; \quad \frac{dW}{dt} = 400\pi \frac{dy}{dt}$$

$$400\pi \frac{dy}{dt} = 9\pi$$

$y$  is increasing at  $\frac{9}{400} \text{ ft/min}$

5)

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let  $h$  be the depth of the coffee in the pot, measured in inches, where  $h$  is a function of time  $t$ , measured in seconds. The volume  $V$  of coffee in the pot is changing at the rate of  $-5\pi\sqrt{h}$  cubic inches per second. (The volume  $V$  of a cylinder with radius  $r$  and height  $h$  is  $V = \pi r^2 h$ .)



- (a) Show that  $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ .
- (b) Given that  $h = 17$  at time  $t = 0$ , solve the differential equation  $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$  for  $h$  as a function of  $t$ .
- (c) At what time  $t$  is the coffeepot empty?

(a)  $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

(b)  $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}} dh = -\frac{1}{5} dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

(c)  $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

$$3 : \begin{cases} 1 : \frac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \frac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$$

$$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

1 : answer

6)

(a)  $V = \pi r^2 h = \pi(1)^2 h = \pi h$

$$\left. \frac{dV}{dt} \right|_{h=4} = \pi \left. \frac{dh}{dt} \right|_{h=4} = \pi \left( -\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

(b)  $\frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left( -\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$

Because  $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$  for  $h > 0$ , the rate of change of the height is increasing when the height of the water is 3 feet.

(c)  $\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left( -\frac{1}{20}t + \sqrt{5} \right)^2$$

$$2 : \begin{cases} 1 : \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$$

$$3 : \begin{cases} 1 : \frac{d}{dh} \left( -\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1 : \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration