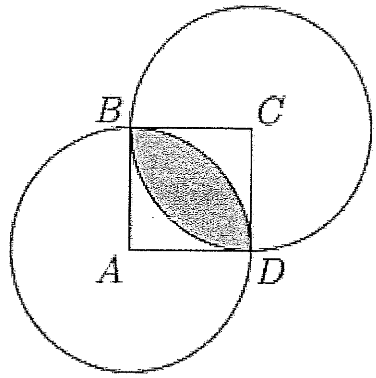
Math Team Problems Thursday September 17, 2015

1.

The square $ABCD$ in the figure has sides of length 1. Both of the circles pass through the points B and D, but one is centered at A and the other at C. What is the area of the intersection of the two circles (the shaded region in the figure)?

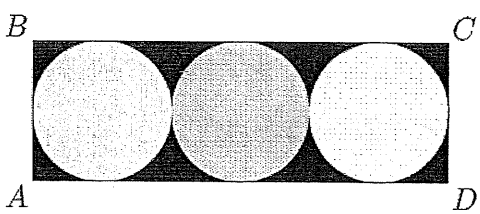


(A) $\frac{\pi}{4}$ (B) $1-\frac{\pi}{4}$ (C) $\frac{\pi}{2}-1$

(D) $\frac{3\pi}{4}-1$ (E) $\frac{1}{3}$

2.

Three circles of the same radius are inscribed in a rectangle as shown below. Given $BC = 5$, find the length of AB. Give your answer as a fully simplified fraction.



3.

Find the value of $1-2+3-4+\cdots+2011$.

4.

37. A train leaves from South Detroit going to Anywhere at 8:00 AM traveling at 20 mph. A second train leaves from South Detroit at 10:00 AM following the first train at 30 mph. How many miles does the first train travel before the second train catches up?

5.

Compute the sum of the infinite geometric series $3, 2, \frac{4}{3}, \frac{8}{9}, \ldots$

6.

If $mn = 1$ and $m^{2a} = n^{2+a}$, what is a

7.

10. What is the probability that an integer in the set $\{1, 2, 3, ..., 100\}$ is divisible by 2 but not divisible by 3?

8.

If P is the midpoint of an edge of a cube with volume 216, find the maximum distance between P and a vertex of the cube.

9.

Find all the x-intercepts of the graph of:

$|x+3| + |y-1| = 6$

10.

If **equilateral triangle** ΔBAT has side length 8, **square** $MASK$ has diagonal length 8, **circle** O has diameter 7, and **regular hexagon** $GOBLIN$ has side length 3, which figure will have the **largest area?**

Recall:

The area of an equilateral triangle can be found as $Area = \frac{s^2\sqrt{3}}{4}$

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1.

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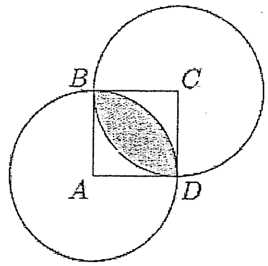
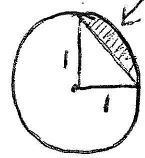




Quarter circle Area − triangle Area

$= \frac{1}{4}(\pi r^2) - \frac{1}{2}bh$

$= \frac{\pi}{4}(1)^2 - \frac{1}{2}(1)(1) = \frac{\pi}{4} - \frac{1}{2}$

$= \frac{\pi - 2}{4}$

double the area

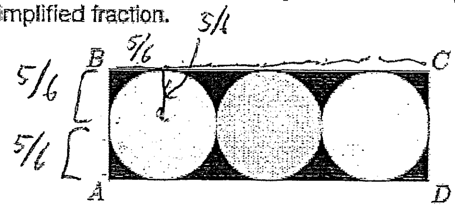
$2\left[\frac{\pi-2}{4}\right] = \frac{\pi - 2}{2}$

$= \boxed{\frac{\pi}{2} - 1}$

2.

Three circles of the same radius are inscribed in a rectangle as shown below. Given $BC = 5$, find the length of AB. Give your answer as a fully simplified fraction.

radius $= \frac{5}{6}$



$\frac{5}{6} + \frac{5}{6} = \frac{10}{6} = \boxed{\frac{5}{3}}$

3.

We can pair up the numbers

$$(1-2)+(3-4)+(2009-2010)+2011,$$

so there are 1005 pairs and each pair becomes -1. The expression becomes: $(-1(1005)+2011 = 2011 - 1005 = \boxed{C)\ 1006}$.

4.

If x is the time the second train travelled for, the distance it travels is $30x$. The distance the first train travels is $20(x+2)$ to account for the 2 hour head start. Setting the distances equal, $30x = 20(x+2)$. Solving yields 4 hours, so they travel $\boxed{A)\ 120}$ miles before meeting.

5.

The sum is $\frac{3}{1-\frac{2}{3}} = \boxed{E)\ 9}$.

6.

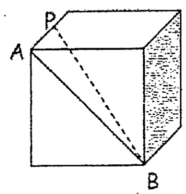
We know $m = \frac{1}{n}$, so $n^{2+a+2a} = 1$. Thus, $3a + 2 = 0$, so $a = \boxed{C)\ \frac{-2}{3}}$.

7.

10. Integers that are divisible by 2 are 50 of the total 100 integers, but integers divisible by 2 and 3 are divisible by 6. There are 17 such multiples of 6, so there are $50 - 16 = 34$ desired integers. The probability is $\boxed{C)\ \frac{17}{50}}$.

8.

<u>*Varsity Group Ciphering Set #3 - Question #4:*</u>



Consider $\triangle PAB$. $PA = 3$ and $AB = 3\sqrt{2}$. Since $\angle PAB$ is A right angle, $PB = \sqrt{9+18} = \sqrt{27} = 3\sqrt{3}$

Answer: $3\sqrt{3}$

9.

In order to find x intercepts, let y = 0, so $|x + 3| + |y - 1| = 6 \Rightarrow |x + 3| + |-1| = 6 \Rightarrow$ $|x + 3| = 5$ Therefore $x = 2$ or $x = -8$ **Answer: $x = 2$ or $x = -8$**

Note: Also accept (2, 0) and (-8, 0).

10.

The area of an equilateral triangle can be found as $Area = \frac{s^2\sqrt{3}}{4}$,

so its area $= \frac{8^2\sqrt{3}}{4} = \frac{64\sqrt{3}}{4} = 16\sqrt{3} \approx 16(1.7) \approx$ a little less than 32.

The area of a square can be found as $Area = \frac{1}{2} d_1 d_2$,

so its area $= \frac{1}{2}(8)(8) = \frac{1}{2}(64) = 32$.

The area of a circle can be found as $Area = \pi r^2$,

So its area $= \pi\left(\frac{7}{2}\right)^2 = \frac{49\pi}{4} = (12.25)(3.14) \approx$ a little more than 36.

The area of a regular hexagon can be found by dividing the hexagon up into 6 congruent equilateral triangles, which means its area can be found as $Area = 6 \cdot \frac{s^2\sqrt{3}}{4}$,

so its area $= 6 \cdot \frac{3^2\sqrt{3}}{4} = \frac{27\sqrt{3}}{2} = (13.5)(1.7) \approx$ a little less than 27.

Thus, $\frac{27\sqrt{3}}{2} < 16\sqrt{3} < 32 < \frac{49\pi}{4}$,

so the areas line up as: hexagon GOBLIN < triangle BAT < square MASK < circle O

Thus, circle O has the largest area.

Answer: circle O