Solutions **Ciphering Round Aug 27 2015 (thurs)**

1. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

   Answer: 15

   Solution: Let d be the length of one lap in miles. Then he needs to complete the four laps in $\frac{4d}{10} = \frac{2d}{5}$ hours. He has already spent $\frac{3d}{9} = \frac{d}{3}$ hours on the first three laps, so he has $\frac{2d}{5} - \frac{d}{3} = \frac{d}{15}$ hours left. Therefore, he must maintain a speed of $\boxed{15}$ mph on the final lap.

2. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

   Answer: 10/11

   Solution: After removing x from 10, and then increasing that amount by 10%, we must end up with at least the amount we started with, 10 pounds. That is, the maximum value of x must satisfy $\frac{11}{10}(10 - x) = 10$. Solving for x, we get that $x = \boxed{10/11}$.

3.

Given that $f(x) + 2f(8-x) = x^2$ for all real x, compute $f(2)$.

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Answer: 68/3

Solution: Substituting $x = 2$, we get that $f(2) + 2f(6) = 4$. Substituting $x = 6$, we get that $f(6) + 2f(2) = 36$. Solving for $f(2)$ and $f(6)$ gives us that $f(6) = -28/3$ and $f(2) = \boxed{68/3}$.

4.

Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.

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Answer: $\frac{1+\sqrt{13}}{2}$

Solution: Note that $x^4 - x^3 - 5x^2 + 2x + 6 = (x^4 - 5x^2 + 6) - x(x^2 - 2) = (x^2 - 2)(x^2 - 3) - x(x^2 - 2) = (x^2 - 2)(x^2 - x - 3)$. The two largest candidate roots are therefore $\sqrt{2}$ and $\frac{1+\sqrt{13}}{2}$. Note that $\sqrt{13} > 3$, so $\frac{1+\sqrt{13}}{2} > 2 > \sqrt{2}$, so therefore the largest root is $\boxed{\frac{1+\sqrt{13}}{2}}$.

5.

Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x+13}} = 13$.

Answer: 546/5

Solution: Observe that $f(a) = \sqrt[3]{20x+a}$ is an increasing function in a, so the only way that $f(f(a)) = a$ can be true is if $f(a) = a$. Solving $\sqrt[3]{20x+13} = 13$, we obtain $x = \boxed{546/5}$.

6.

In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

Answer: $\sqrt{51}$

Solution: Let $m\angle A = x$ and $m\angle B = y$. Note that we have two pairs of isosceles triangles, so $m\angle A = m\angle ACD$ and $m\angle B = m\angle BCD$. Since $m\angle ACD + m\angle BCD = m\angle ACB$, we have

$$180^\circ = m\angle A + m\angle B + m\angle ACB = 2x + 2y \implies m\angle ACB = x + y = 90^\circ.$$

Since $\angle ACB$ is right, we can use the Pythagorean Theorem to compute BC as

$$\sqrt{10^2 - 7^2} = \boxed{\sqrt{51}}.$$

For a shortcut, note that D is the circumcenter of ABC and lies on the triangle itself, so it must lie opposite a right angle.

7.

Peter is chasing after Rob. Rob is running on the line $y = 2x + 5$ at a speed of 2 units a second, starting from the point $(0, 5)$. Peter starts running t seconds after Rob, running at 3 units a second. Peter also starts at $(0, 5)$, and catches up to Rob at the point $(17, 39)$. What is the value of t?

Answer: $\dfrac{17\sqrt{5}}{6}$

Solution: Rob runs a distance of $\sqrt{17^2 + 34^2} = 17\sqrt{5}$ units. Therefore, Rob runs for a total of $\dfrac{17\sqrt{5}}{2}$ seconds. Peter must therefore run a total of $\dfrac{17\sqrt{5}}{2} - t$ seconds, and we know that $3\left(\dfrac{17\sqrt{5}}{2} - t\right) = 17\sqrt{5}$. Solving for t, we get $t = \boxed{\dfrac{17\sqrt{5}}{6}}$.

8.

An isosceles right triangle is inscribed in a circle of radius 5, thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.

Answer: $\dfrac{25\pi}{2} - 25$

Solution:

We use the fact that the hypotenuse of any right triangle that is inscribed in a circle is actually a diameter of the circle.

The area of the circle is 25π. The hypotenuse creates two semicircles of area $\dfrac{25\pi}{2}$ each. The legs divide one of these semicircles into three regions, including a right triangle with area $\dfrac{(5\sqrt{2})^2}{2} = 25$. The other two regions sum to $\dfrac{25\pi}{2} - 25$. Since $25 > \dfrac{25\pi}{2} - 25$, the sum of the areas of the two smallest regions is $\boxed{\dfrac{25\pi}{2} - 25}$.

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5. Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x + 13}} = 13$.

6. In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

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