(Calculator)

Scoring Guideline

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

2 3 5

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

(a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

(b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

(c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

(d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

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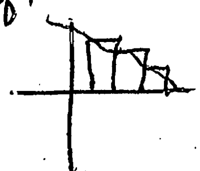
2 a) Volume $\displaystyle\int_0^{10} A(h)\,dh \approx 2(50.3) + 3(14.4) + 5(6.5) = \boxed{176.3 \text{ ft}^3}$

2 b) Since $A(h)$ is decreasing, left Riemann Sum will overestimate the volume of tank.



2 c) $\displaystyle V(h) = \int_0^{10} f(h)\,dh = \int_0^{10} \frac{50.3}{e^{0.2h}+h}\,dh = \boxed{101.325 \text{ ft}^3}$

3 d) $\displaystyle V = \int_0^h f(x)\,dx$

$\displaystyle \frac{dV}{dt} = \frac{d}{dt}\int_0^h f(x)\,dx$

Apply SFTC $\displaystyle \frac{d}{dx}\int^{p(x)} f(t)\,dt = f(p(x))\cdot p'(x)$

$\displaystyle \frac{dV}{dt} = f(h)\cdot\frac{dh}{dt}$

$\begin{cases} \dfrac{dh}{dt} = 0.26 \text{ ft/min} \\ h = 5 \end{cases}$

$\displaystyle \frac{dV}{dt} = \frac{50.3}{e^{0.2h}+5}\cdot\frac{dh}{dt}$

$\displaystyle \frac{dV}{dt} = \frac{50.3}{e^{0.2(5)}+5}(0.26) = \boxed{1.694 \text{ ft}^3/\text{min}}$

2)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$$2 : \begin{cases} 1 : \dfrac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

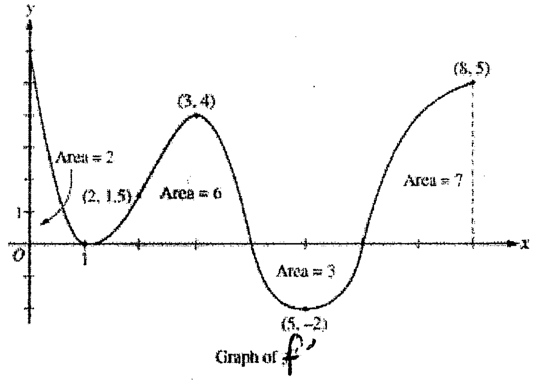
$$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.
$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$
An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$$3 : \begin{cases} 1 : g^{-1}(2) \\ 1 : \left(g^{-1}\right)'(2) \\ 1 : \text{tangent line equation} \end{cases}$$

3) (Non-calculator)

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.



(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

$\boxed{1}$ a) $\nearrow$ $+$ on $(0,1)$, $\nearrow$ $+$ on $(1,4)$ … $\searrow$ $-$ on $(4,6)$, $\nearrow$ $+$ on $(6,8)$; marks: 0, 1, 4, 6, 8

Relative minimum at $x=6$ since $f'(x)$ changes from $-$ to $+$.

$\boxed{3}$ b) Absolute minimum can occur at relative minimum or endpoints (EVT)

Find points at $x = 0, 6, 8$ Given: $f(8) = 4$

$* \; f(b) = f(a) + \int_a^b f'(x)\,dx$

$f(0) = f(8) + \int_8^0 f'(x)\,dx \rightarrow f(0) = f(8) - \int_0^8 f'(x)\,dx = 4 - (2+6-3+7) = \boxed{-8}$

$f(6) = f(8) + \int_8^6 f'(x)\,dx \rightarrow f(6) = f(8) - \int_6^8 f'(x)\,dx = 4 - (7) = \boxed{-3}$

$f(8) = \boxed{4}$

Absolute minimum value is -8 on the closed interval $[0,8]$

$\boxed{2}$ c) $f''(x)$: $\cap$ $-$ on $(0,1)$, $\cup$ $+$ on $(1,3)$, $\cap$ $-$ on $(3,5)$, $\cup$ $+$ on $(5,8)$; marks: 0, 1, 3, 5, 8

$f'(x)$: $\nearrow$ $+$ on $(0,1)$, $\nearrow$ $+$ on $(1,4)$, $\searrow$ on $(4,6)$, $\nearrow$ on $(6,8)$; marks: 0, 1, 4, 6, 8

The graph of f is concave down and increasing on $0<x<1$ and $3<x<4$ b/c $f'(x)$ is decreasing and positive in these intervals

$\boxed{3}$ d) $g(x) = [f(x)]^3$

*chain rule $\rightarrow$ $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$g'(3) = 3[f(3)]^2 \cdot f'(3)$

$f'(3) = 4$

$g'(3) = 3\left[\frac{-5}{2}\right]^2 \cdot 4$

$g'(3) = 3 \cdot \left(\frac{25}{\cancel{4}}\right) \cdot \cancel{4} = 75$

$\boxed{g'(3) = 75}$

LAB #

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	−6	3	2	8
2	2	−2	−3	0
3	8	7	6	2
6	4	5	3	−1

4) The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

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3 a) * To write tangent line equation, find point and slope:

point: $K(x) = f(g(x))$

$K(3) = f(g(3))$

$= f(6)$

$K(3) = 4$

slope: * Use chain rule to find $K'(x)$ first.

$K(x) = f[g(x)]$

$K'(x) = f'[g(x)] \cdot g'(x)$

$K'(3) = f'[g(3)] \cdot g'(3)$

$K'(3) = f'(6) \cdot g'(3)$

$= f'(6) \cdot g'(3)$

$= 5 \cdot 2$

$K'(3) = 10$

$\boxed{\text{Tangent line equation: } y - 4 = 10(x-3)}$

3 b) * Use quotient rule first to find $h'(x)$. Then find $h'(1)$

$h'(x) = \dfrac{g'(x)f(x) - g(x)f'(x)}{f(x)^2}$

$h'(1) = \dfrac{g'(1)f(1) - g(1)f'(1)}{f(1)^2} = \dfrac{8(-6) - 2(3)}{(-6)^2}$

$h'(1) = \dfrac{-54}{36} = \boxed{\dfrac{-3}{2}}$

3 c) * Go through u-substitution first to handle the $(2x)$

$\int_1^3 f''(2x)\,dx$

$u = 2x \quad dx = \dfrac{du}{2}$

$\dfrac{du}{dx} = 2$

$\int f''(u) \cdot \dfrac{du}{2}$

$= \dfrac{1}{2}\int f''(u)\,du$

* Apply FFTC $\int_a^b f'(x)\,dx = f(b) - f(a)$

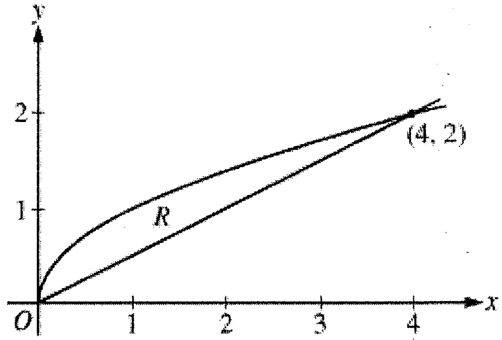
$= \dfrac{1}{2} f'(2x)\Big]_1^3 = \dfrac{1}{2} \cdot f'(2\cdot 3) - \dfrac{1}{2} f'(2\cdot 1)$

$= \dfrac{1}{2} f'(6) - \dfrac{1}{2} f'(2)$

$= \dfrac{1}{2}(5) - \dfrac{1}{2}(-2) = \boxed{7/2}$

5)

Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the vertical line $x=-3$

$\boxed{3 pts}$ (a) Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

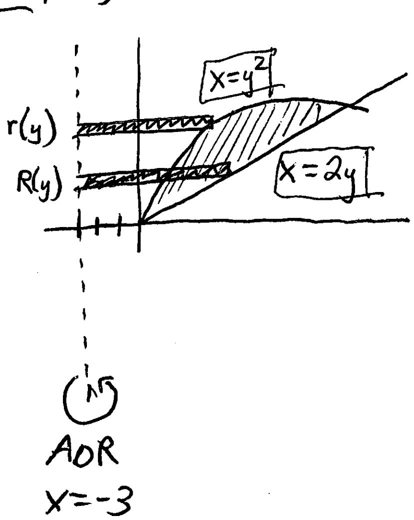
$\boxed{3 pts}$ (b) Volume $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big|_{x=0}^{x=4} = \frac{8}{15}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

$\frac{4^2}{2} - \frac{2(4)^{5/2}}{5} + \frac{4^3}{12} - (0 - 0 + 0) = \boxed{\frac{8}{15}}$

$\boxed{3 pts}$ c) AOR: $x=-3$



*washer method
Right/Left

$x = \underline{2y}$

$x = \underline{y^2}$

$R(y) = 2y-(-3)$

$r(y) = y^2 - -3$

*bounds: $2y = y^2$
$2y - y^2 = 0$

$y(2-y) = 0$
$y = 0, 2$

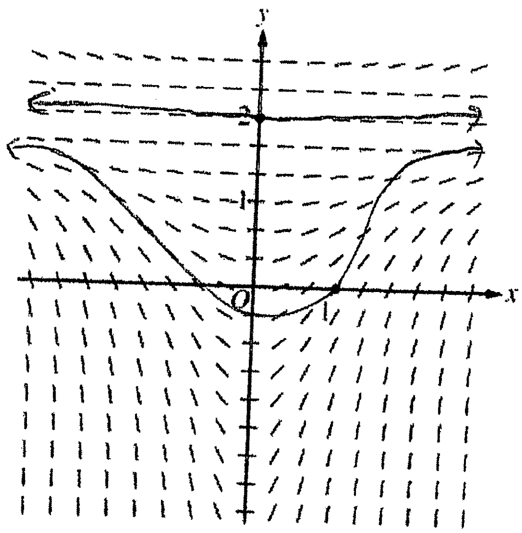
$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 \, dy$

$$\boxed{V = \pi \int_0^2 [2y+3]^2 - [y^2+3]^2 \, dy}$$

AB #6

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

2 (a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



c) continued...

$$\frac{-1}{y-2} = \frac{2x^2+1}{6}$$

$$(y-2)(2x^2+1) = -6$$

$$y-2 = \frac{-6}{2x^2+1}$$

$$\boxed{y = \frac{-6}{2x^2+1} + 2}$$

check with $(1,0)$ ✓

2 (b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

5 (c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

b) *equation of tangent line: find point: $(1,0)$

slope: $\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{1}{3}(1)(0-2)^2 = \frac{4}{3}$

$$y - y_1 = m(x - x_1)$$
$$y - 0 = \frac{4}{3}(x-1)$$

$$y = \frac{4}{3}(x-1)$$
$$y(0.7) = \frac{4}{3}(0.7-1) = \frac{4}{3}(-0.3) = \boxed{-0.4}$$

c) $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$

$$dy = \frac{1}{3}x(y-2)^2\,dx$$

$$\int \frac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx$$

$u = y-2$

$\frac{du}{dy} = 1$

$dy = du$

$$\int \frac{du}{u^2} = \int \frac{1}{3}x\,dx$$

$$\int u^{-2}\,du$$

$$\frac{u^{-1}}{-1} = \frac{1}{3}\frac{x^2}{2} + C$$

$$\frac{-1}{y-2} = \frac{x^2}{6} + C$$

*plug in $(1,0)$

$$\frac{-1}{0-2} = \frac{1}{6} + C$$

$$\frac{1}{2} = \frac{1}{6} + C \qquad C = \frac{1}{3}$$

$$\frac{-1}{y-2} = \frac{x^2}{6} + \frac{1}{3}$$

$$\frac{-1}{y-2} = \frac{2x^2+1}{6}$$

continued...