The Georgia Council of Teachers of Mathematics

Presents

The 40th Annual
STATE MATHEMATICS TOURNAMENT

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Hosted by GCTM and Middle Georgia State University

Middle Georgia State University

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Instructions:

1. Do NOT open this test booklet until told to do so.
2. Your ID number given to you by your sponsor MUST be entered correctly on your answer sheet, or your test will not be scored. Write your name and school on the answer sheet. If you are competing for ARML consideration and are not part of your school's four-person team, please write "ARML" next to your school.
3. You will need a calculator for this test. Any calculator that does NOT have a symbolic algebra system is allowed.
4. This is a 90-minute, 50-problem examination. The first 45 questions are multiple choice with five responses for each question. You are to select the one best answer for each question and darken the letter corresponding to that answer on the answer sheet provided. In order for your answer sheet to be graded correctly, be sure that you darken only one answer per problem, that the space is darkened completely, and that all erasures are complete.
5. Write your answers to problems 46–50 on the back of your answer sheet. Label them well and write clearly! The graders will not make guesses as to whether you wrote "4" or "9". Unclear answers will be counted as incorrect.
6. The test monitor will announce when 30 minutes remain and when 5 minutes remain. When time is called, place your pencils down.
7. Your score will be computed by the following formula:

   $$\text{Score} = 5 \times (\text{number correct}) + (\text{number of multiple-choice left blank}).$$

   You will receive no points for blanks on problems 46–50. The maximum possible score is 250 and there is no penalty for incorrect answers.

1. In Mathematicity, the capitol of Mathland, on the famous boardwalk Hilbert Street, there are six exclusive hotels: the Hotel Abel, the Brahmagupta Ballroom, the Cauchy Grand Hotel, the Descartes Inn, Euler Suites, and Fermat Towers. Each hotel has 12 rooms available for booking for the Math Contest Problem Writers Convention. Compute the minimum number of rooms needed to be booked to guarantee that Hotel Abel gets at least 2 rooms reserved.

   (A) 2 (B) 7 (C) 14 (D) 62 (E) 74

2. A rectangle with sides x and y has a diagonal of length 12. Given that $y - x = 10$, compute xy.

   (A) $\frac{69}{4}$ (B) 22 (C) 65 (D) $\frac{143}{2}$ (E) 135

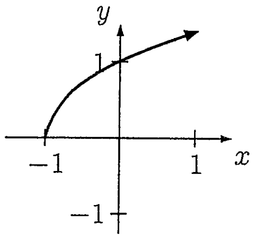
3. Compute the area of a triangle whose side lengths are 5, 6, and 7.

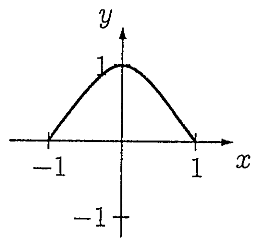
   (A) $5\sqrt{3}$ (B) $6\sqrt{6}$ (C) 15 (D) $12\sqrt{2}$ (E) $12\sqrt{3}$

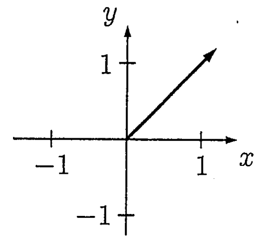
4. Suppose $f(x) = 3x^2 - 5x + 2$. Compute the sum of the zeros of $f(x-2)$.

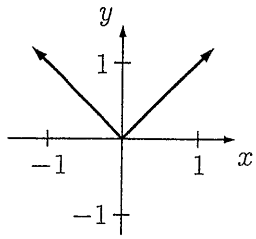
   (A) $-\frac{1}{3}$ (B) 2 (C) 3 (D) $\frac{11}{3}$ (E) $\frac{17}{3}$

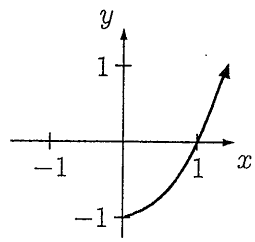
5. Given the functions $f(x) = x^2 + 1$, $g(x) = x - 2$, and $h(x) = \sqrt{x+1}$, which of the following is the graph of $y = h(g(f(x)))$?

(A) 

(B) 

(C) 

(D) 

(E) 

6. Compute the sum of the solutions to the equation $5^{x^2} = 25^{x+1}$.

(A) -2 (B) $\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 2

7. Determine the domain of the function $f(x) = \dfrac{2}{\sqrt{x} - \sqrt{4-x}}$.

(A) all real numbers (B) $[1, 3]$ (C) $(0, 4)$ (D) $[0, 4]$ (E) $[0, 2) \cup (2, 4]$

8. Suppose $r = a/b$, with a and b distinct nonzero integers. Which of the following is equivalent to $a/(a-b)$?

(A) $r-1$ (B) $\frac{r+1}{r}$ (C) $\frac{r-1}{r}$ (D) $\frac{r+1}{r-1}$ (E) $\frac{r}{r-1}$

9. Compute the number of zeroes at the end of 105!.

(A) 21 (B) 23 (C) 24 (D) 25 (E) 27

10. Compute the sum of the solutions to the equation $\sin^2\theta - \sin\theta = 1$, where θ is in radians, on the interval $[0, 2\pi)$.

(A) π (B) $\frac{6\pi}{5}$ (C) 3π (D) $\frac{9\pi}{2}$ (E) 5π

11. Evaluate $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{2016}$.

(A) 1 (B) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (C) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (D) $1008 + 1008i\sqrt{3}$ (E) $1008\sqrt{3} - 1008i$

12. Given that $|x| \neq |y|$, simplify $\dfrac{x^2 - xy + y^2}{x^3 + y^3} - \dfrac{x^2 + xy + y^2}{x^3 - y^3}$.

(A) $2x$ (B) $\dfrac{1}{x^6 - y^6}$ (C) $\dfrac{-2y}{x^4 - x^2y^2 + y^4}$ (D) $\dfrac{2x}{x^2 - y^2}$ (E) $\dfrac{2y}{y^2 - x^2}$

13. The polynomial $P(x) = 2x^4 - 3x^3 + ax + b$, where a and b are real numbers, can be divided evenly by $x^2 - 4$. Compute $a + b$.

(A) -20 (B) -12 (C) 8 (D) 12 (E) 20

14. Compute the units digit of the sum of the coefficients of the expansion of $(8 + x)^{2016}$.

(A) 1 (B) 2 (C) 6 (D) 8 (E) 9

15. Compute the sum of all possible values of k so that the piecewise function f, defined below, is continuous.

$$f(x) = \begin{cases} (k+1)x + 10 & \text{for } x \leq k \\ 2x^2 + 3k - 5 & \text{for } x > k \end{cases}$$

(A) -6 (B) -2 (C) 1 (D) 2 (E) 6

16. What is the probability of rolling a fair die four times and observing the 5 face up at least twice?

(A) $\frac{5}{324}$ (B) $\frac{25}{1296}$ (C) $\frac{19}{324}$ (D) $\frac{19}{144}$ (E) $\frac{125}{324}$

17. The counting numbers are arranged in rows with one number in the first row, two numbers in the second row, three numbers in the third row, and so forth, as shown below.

		columns					
		1	2	3	4	5	...
	1	1					
	2	2	3				
rows	3	4	5	6			
	4	7	8	9	10		
	5	11	12	13	14	15	
	⋮						⋱

What is the sum of the row number and column number for the entry 2016? (For example, in the diagram above, the entry 9 is in row 4 column 3 so the sum is 7.)

(A) 63 (B) 64 (C) 65 (D) 125 (E) 126

18. Three marbles are to be drawn from among three urns. The first will be randomly drawn from an urn that contains exactly 8 blue marbles and 2 yellow marbles. The second will be randomly drawn from an urn that contains exactly 8 red marbles and 2 yellow marbles. The third will be randomly drawn from an urn that contains exactly 8 green marbles and 2 yellow marbles. To the nearest hundredth, what is the probability that at least one yellow marble will be drawn?

(A) 0.33 (B) 0.49 (C) 0.50 (D) 0.51 (E) 0.67

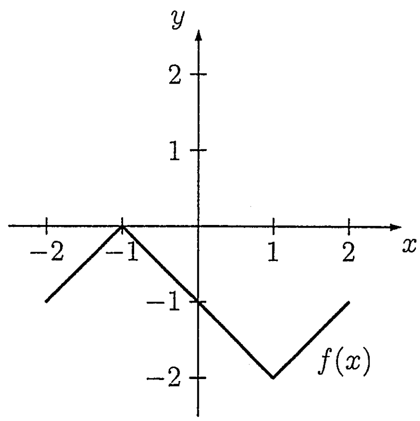
19. Dr. Beaker made a mistake preparing the HCl solution for this week's lab. Instead of preparing 5 liters of 2 M HCl, she prepared 5 liters of 1 M HCl. (A 1 M solution of hydrochloric acid contains 1 mole of HCl per liter of solution.) If she uses all of the 1 M HCl solution she has prepared, how many liters of 12 M stock solution should she add to get a 2 M HCl solution?

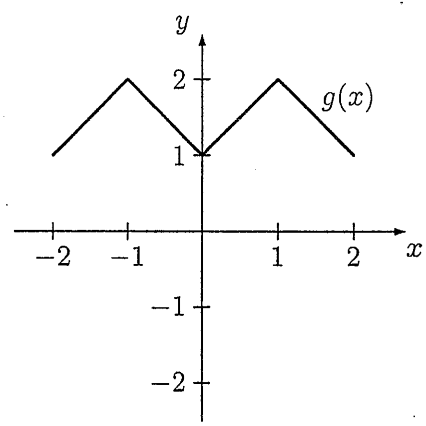
(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 1 (E) 2

20. Suppose $\sin A + \cos B = 1.5$ and $\sin A - \cos B = 0.3$, where A and B are both first quadrant angles. The exact value of $\sin(A+B)$ can be written as $\frac{m+n\sqrt{p}}{q}$, where m, n, p, and q are positive integers, m and q are relatively prime, and p is a prime number. Compute $m+n+p+q$.

(A) 58 (B) 86 (C) 100 (D) 176 (E) None of these

21. The graphs of functions f and g, for $-2 \le x \le 2$, are shown below.



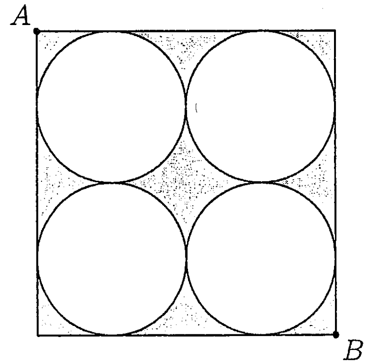


Which of the following statements is/are true?

I. $g(x) = -f(|x|)$
II. $g(x) = -|f(x)|$
III. $g(x) = |f(|x|)|$

(A) I only (B) II only (C) III only (D) I and II only (E) I and III only

22. In the figure below, four congruent circles each of radius 1 unit are packed into a square. If a particle must travel a path within the shaded area or along its boundary, compute the length of the shortest path from A to B.

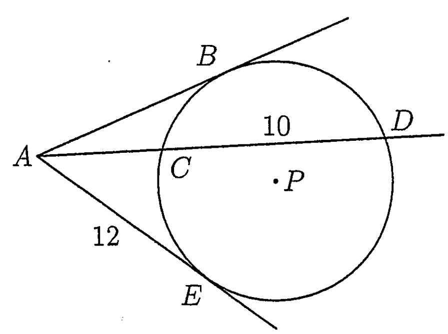


(A) $2+\frac{3\pi}{2}$ (B) $4+\pi$ (C) $4\left(\sqrt{2}-1\right)+2\pi$ (D) 8 (E) $4+4\sqrt{2}$

23. A circle has diameter $\overline{AB}$ with A and B on the circle. Point C is on the circle, but at neither A nor B. A circle with center D is inscribed in $\triangle ABC$. Compute the measure of $\angle ADB$ in degrees.

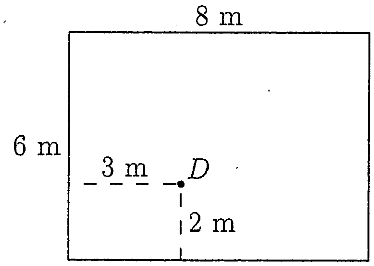
(A) 90° (B) 105° (C) 135° (D) 150° (E) Cannot be determined

24. In the diagram below, $\overline{AE}$ and $\overline{AB}$ are tangent to circle P, and a line from A intersects circle P at C and D. Given that $AE = 12$ and $CD = 10$, compute $AC \cdot AB$.



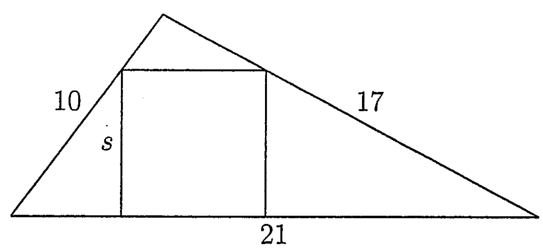
(A) 24 (B) 48 (C) 72 (D) 96 (E) Cannot be determined

25. An evil mutated alien Kaled from the planet Skaro is trapped! The alien is trapped in a room which is impervious to laser-beam weapons! The rectangle below represents the room with dimensions 6×8 meters. The Kaled is positioned 2 meters from the 8-meter wall and 3 meters from the 6-meter wall. It fires a laser beam weapon from point D. The laser beam reflects off of three walls (avoiding corners) before returning to point D and exterminating itself. What is the minimum length traversed by the beam?



(A) $6\sqrt{5}$ (B) $4\sqrt{13}$ (C) $2\sqrt{61}$ (D) $4\sqrt{20}$ (E) $5\sqrt{65}$

26. A square is inscribed in a triangle with side lengths 10, 17, and 21, such that two vertices of the square are on the long side of triangle and one vertex is on each of the other sides, as shown below. What is the side length of the square, rounded to the nearest hundredth?

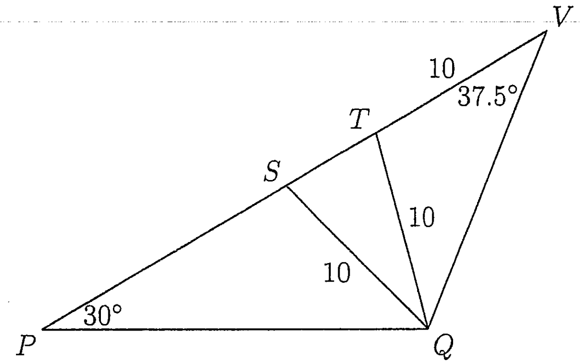


(A) 5.28 (B) 5.79 (C) 5.93 (D) 6.21 (E) 6.36

27. Your little brother rips a bunch of consecutive pages out of your favorite mathematics textbook. Given that the pages are printed back-to-back, and the sum of the page numbers which were ripped out is 1616, what was the number of the first page ripped out?

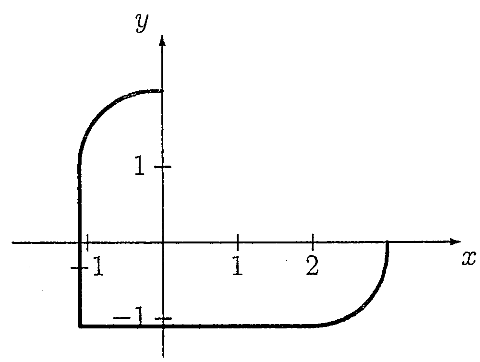
(A) 16 (B) 35 (C) 66 (D) 68 (E) 101

28. In the figure below, $\angle P = 30°$, $\angle V = 37.5°$, and $SQ = TQ = TV = 10$. Compute PQ.

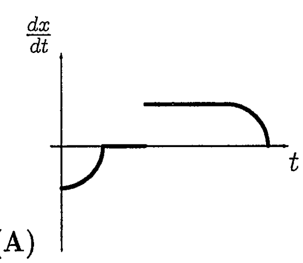


(A) $5\sqrt{3}+5$ (B) $10\sqrt{2}$ (C) $10\sqrt{3}$ (D) $5\sqrt{2}+5\sqrt{6}$ (E) $20\sqrt{2}$

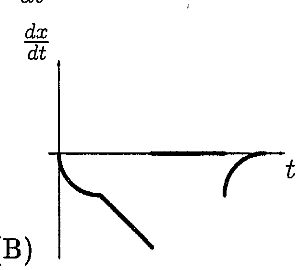
29. The graph below describes the path of a particle. The particle starts at $(0,2)$ and moves along the path shown at a constant speed, ending at $(3,0)$.



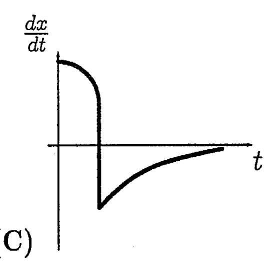
Which graph best describes $\frac{dx}{dt}$ for the particle during this journey?



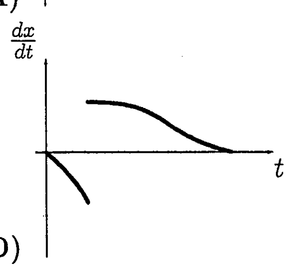
(A)



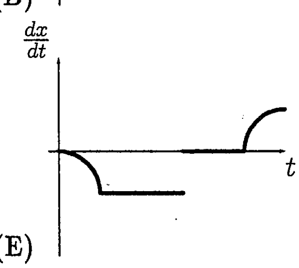
(B)



(C)



(D)



(E)

30. In space, a vector from point B to point A is given by $\langle 1, -2, 3\rangle$, and a vector from point B to point C is given by $\langle 2, 1, -3\rangle$. Compute $\sin\angle ABC$.

(A) 0 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{115}}{14}$ (D) $\frac{\sqrt{3}}{2}$ (E) 1

31. Let k be the smallest natural number such that $k! + (k-1)!$ is divisible by 256. Compute the remainder when k is divided by 5.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

32. The sum

$$\sum_{n=1}^{\infty} \frac{\pi n - 1}{4^n} = \frac{\pi - 1}{4} + \frac{2\pi - 1}{16} + \frac{3\pi - 1}{64} + \frac{4\pi - 1}{256} + \cdots$$

can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are integers and a and c are relatively prime. Compute $a + b + c$.

(A) 16 (B) 19 (C) 20 (D) 22 (E) 25

33. Josh and Marcy are playing a dice rolling game. They each roll one fair six-sided die. Whoever rolls the higher number wins. If they roll the same number, there is no winner. If Josh wins, what is the probability that he rolled a 5 or a 6?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{5}{12}$ (E) $\frac{3}{5}$

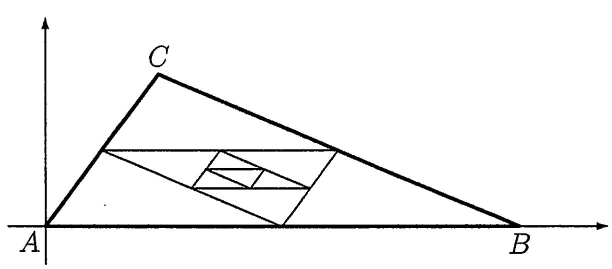
34. An arithmetic sequence and a geometric sequence have the same first and fourth terms. If the second term of the arithmetic sequence is 215 and its fifth term is 500, what is the third term of the geometric sequence?

(A) 120 (B) 240 (C) 270 (D) 310 (E) 405

35. Compute the area of the largest triangle, rounded to the nearest tenth of a square inch, that will fit in a box measuring 6 inches by 8 inches by 3 inches.

(A) 24 (B) 25.6 (C) 26.8 (D) 28.3 (E) 29.2

36. Let $\triangle ABC$ have vertices $A(0,0)$, $B(20,0)$, and $C(4,18)$. As shown below, the midpoints of the sides are connected to form a smaller inner triangle. Then, the midpoints of the sides of this smaller triangle are connected in a similar way to form an even smaller triangle. This process is continued *ad infinitum.* Let (x, y) be the coordinates of the only point contained in all such constructed inner triangles. Compute $x + y$.



(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

37. The function $f(\theta)$ is defined as the following infinite expression.

$$f(\theta) = -\sin(-\theta) + \cos^2(90^\circ - \theta) - \frac{1}{\csc^3(-\theta)} + \frac{1}{\sec^4(90^\circ - \theta)} - \sin^5(-\theta) + \cos^6(90^\circ - \theta) - \frac{1}{\csc^7(-\theta)} + \frac{1}{\sec^8(90^\circ - \theta)} + \cdots$$

Compute $f(60^\circ)$.

(A) 0 (B) 1 (C) $3\sqrt{2}+2$ (D) $2\sqrt{3}+3$ (E) undefined

38. Evaluate the base-2 number 0.110110110110... as a reduced fraction in base-10.

(A) $\frac{2}{3}$ (B) $\frac{4}{5}$ (C) $\frac{6}{7}$ (D) $\frac{8}{9}$ (E) $\frac{10}{11}$

39. The letters of the word AMBIDEXTROUSLY undergo a particular permutation repeatedly. Applying the permutation once results in the word ABMDEITROUXLYS. Applying it again results in AMBEIDROUXTYSL. We apply this permutation again and again and again *ad infinitum*. Let P be the minimum number of permutations which return the letters to their original order. What is the sum of the digits of P?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

40. Mary is at her favorite doughnut shop, Toroidal Treats, trying to decide how many of each type to buy. Her choices are glazed, chocolate, cream, and strawberry doughnuts. If she buys one dozen doughnuts, how many possible choices are there? (She may buy any number of each variety from 0 to 12, and each doughnut of the same variety is assumed to be identical.)

(A) 165 (B) 220 (C) 455 (D) 495 (E) 1365

41. Suppose we say that a real number x is *squishy* whenever $x = a + b\sqrt{10}$ for some integers a and b. Additionally, let us classify a squishy number u as *twonky* if and only if the quotient x/u is squishy for every squishy number x. If u is twonky and can be represented as $u = c + d\sqrt{10}$ where c and d are integers, which of the following could be a value for c?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

42. Suppose n is the smallest positive integer such that $n!$ has 2016 trailing zeroes. Compute the remainder when n is divided by 100.

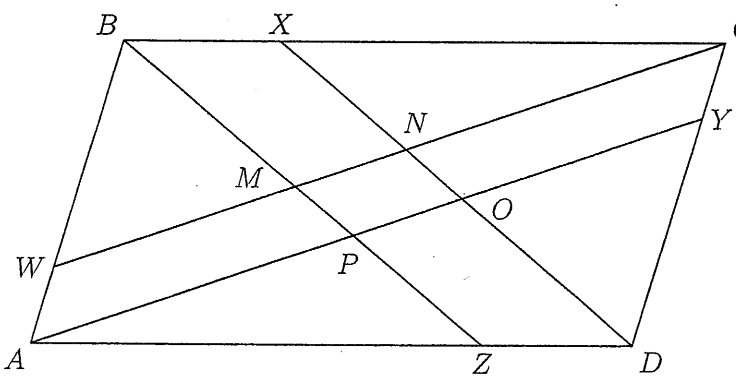
(A) 65 (B) 70 (C) 75 (D) 80 (E) 85

43. Compute $1 + x + x^2 + x^3$ where x is the solution to the equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \cdots}}} = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots}}}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

44. In parallelogram $ABCD$, segments AW, BX, CY, and DZ are one fourth as long as segments AB, BC, CD, and DA, respectively. Find the ratio of the area of parallelogram $MNOP$ to the area of parallelogram $ABCD$.



(A) $\frac{1}{21}$ (B) $\frac{1}{22}$ (C) $\frac{1}{23}$ (D) $\frac{1}{24}$ (E) $\frac{1}{25}$

45. The infinite sum

$$\sum_{n=0}^{\infty} \text{Arctan}\left(\frac{2}{(n+1)^2}\right)$$

can be written in the form $\frac{p}{q}\pi$ where p and q are integers and $\frac{p}{q}$ is a reduced fraction. Compute $p+q$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

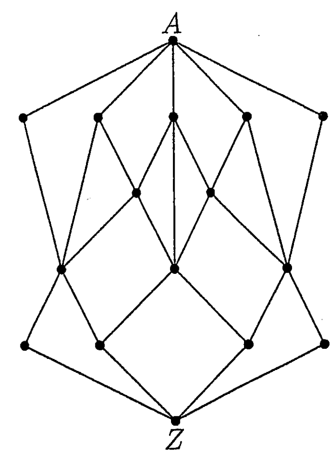
Write your answers to the following problems on the back of your answer sheet.

46. Simplify $\log_2 4 \cdot \log_4 8 \cdot \log_8 16 \cdot \log_{16} 32 \cdot \log_{32} 64 \cdot \log_{64} 128 \cdot \log_{128} 256 \cdot \log_{256} 512 \cdot \log_{512} 1024$.

47. There are five distinct positive integers whose mean is 10. What is the largest value of any one of these numbers?

48. Suppose the real number k is the median of the set $\{9, k, -9, -1, 9, k^2, 1\}$. Then the interval $a \le k \le b$ represents all possibilities for k. Compute $b - a$.

49. The graph at right shows various routes (the edges) between a company's sixteen warehouses (the vertices). From the warehouse labeled A in the north to the warehouse labeled Z in the south, how many possible routes are there from A to Z traveling only in a southward direction?



50. Ten thieves rob a wealthy entrepreneur. The thieves divvy out the spoils so that the first thief gets a tenth of the ill-gotten gains, which happens to be a hoard of identical gold coins. The second thief gets two-tenths of the remainder. The third thief gets three-tenths of that remainder, and so on (the nth thief gets n-tenths of currently remaining gold coins). Compute the smallest number of gold coins possible so that every thief gets a whole number of coins in their share.

THIS IS THE END OF THE TEST. THE REMAINING PAGES ARE FOR SCRATCH WORK.

USE THIS PAPER FOR SCRATCH WORK.