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SOLUTIONS

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GCTM State Tournament 2016 Answers

Individual Ciphering

1. 119
2. 336
3. 25
4. 25
5. 8
6. 8214
7. 3969π
8. 1
9. 225
10. 1

Team Round

1. 9981
2. 2
3. -1
4. 10
5. 38
6. $12 - 3\pi$
7. 780
8. 13
9. 6
10. 51
11. 11
12. 739

1. D
2. B
3. B
4. E
5. D
6. E
7. E
8. E
9. D
10. C
11. A
12. E
13. A
14. A
15. B
16. D
17. E
18. B
19. B
20. C
21. E
22. A
23. C
24. D
25. A
26. B
27. B
28. D
29. A
30. C
31. E
32. A
33. E
34. C
35. D
36. D
37. D
38. C
39. A
40. C
41. B
42. C
43. D
44. E
45. D
46. 10
47. 40
48. 9
49. 26
50. 1562500

1. In Mathematicity, the capitol of Mathland, on the famous boardwalk Hilbert Street, there are six exclusive hotels: the Hotel Abel, the Brahmagupta Ballroom, the Cauchy Grand Hotel, the Descartes Inn, Euler Suites, and Fermat Towers. Each hotel has 12 rooms available for booking for the Math Contest Problem Writers Convention. Compute the minimum number of rooms needed to be booked to guarantee that Hotel Abel gets at least 2 rooms reserved.

(A) 2 (B) 7 (C) 14 (D) 62 (E) 74

> **Answer:** D In the worst case scenario for Niels and Henrik, the managers of Hotel Abel, all of the other rooms of the other hotels (there are $5 \cdot 12 = 60$) would be booked before any of Hotel Abel's rooms were booked. Thus it would take 62 rooms to be sure that they get two.

2. A rectangle with sides x and y has a diagonal of length 12. Given that $y - x = 10$, compute xy.

(A) $\frac{69}{4}$ (B) 22 (C) 65 (D) $\frac{143}{2}$ (E) 135

> **Answer:** B By the Pythagorean Theorem, $x^2 + y^2 = 12^2 = 144$. Squaring the equation $y - x = 10$ gives $y^2 - 2xy + x^2 = 100$. Then $144 - 2xy = 100$, or $xy = (144 - 100)/2 = 22$.

3. Compute the area of a triangle whose side lengths are 5, 6, and 7.

(A) $5\sqrt{3}$ (B) $6\sqrt{6}$ (C) 15 (D) $12\sqrt{2}$ (E) $12\sqrt{3}$

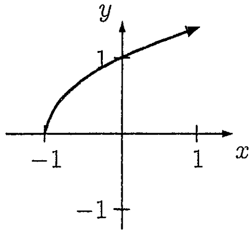
> **Answer:** B We use Heron's formula. The semiperimeter is $s = \frac{1}{2}(5 + 6 + 7) = 9$. The area is then
>
> $$\sqrt{9(9-5)(9-6)(9-7)} = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}.$$

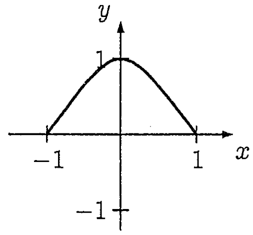
4. Suppose $f(x) = 3x^2 - 5x + 2$. Compute the sum of the zeros of $f(x - 2)$.

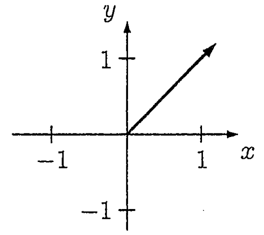
(A) $-\frac{1}{3}$ (B) 2 (C) 3 (D) $\frac{11}{3}$ (E) $\frac{17}{3}$

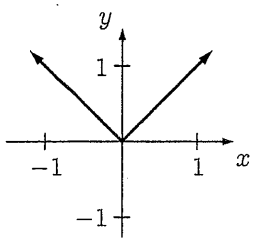
> **Answer:** E First Solution. By Viète's relations, the sum of the roots of $f(x)$ is $\frac{5}{3}$. Translating $f(x)$ to the right 2 units gives us $f(x-2)$, and also adds 2 to each of the roots; hence the sum of the roots of $f(x-2)$ is $\frac{5}{3} + 2 + 2 = \frac{17}{3}$.
>
> Second Solution. We substitute $x - 2$ for every x in the polynomial, and expand. This becomes
>
> $$3(x-2)^2 - 5(x-2) + 2 = 3\left(x^2 - 4x + 4\right) - 5x + 10 + 2 = 3x^2 - 17x + 24.$$
>
> By Viète, the sum of the roots is $\frac{17}{3}$.

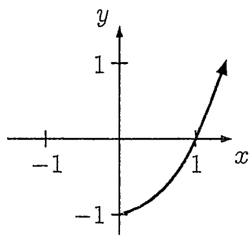
5. Given the functions $f(x) = x^2 + 1$, $g(x) = x - 2$, and $h(x) = \sqrt{x+1}$, which of the following is the graph of $y = h(g(f(x)))$?

(A) 

(B) 

(C) 

(D) 

(E) 

Answer: D We form the composition.

$$y = h(g(f(x))) = h(g(x^2+1)) = h(x^2+1-2) = h(x^2-1) = \sqrt{x^2-1+1} = \sqrt{x^2} = |x|$$

So the composition is the absolute value function, $y = |x|$. (Notice that there are no domain problems.)

6. Compute the sum of the solutions to the equation $5^{x^2} = 25^{x+1}$.

(A) -2 (B) $\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 2

Answer: E Changing the base on the right-hand side yields $5^{2(x+1)} = 5^{2x+2}$. Then we must have $x^2 = 2x+2$, or $x^2 - 2x - 2 = 0$. Hence, the sum of the solutions is 2. (*Note:* Using the quadratic formula yields the two solutions $1 \pm \sqrt{3}$ whose sum is indeed 2.).

7. Determine the domain of the function $f(x) = \dfrac{2}{\sqrt{x} - \sqrt{4-x}}$.

(A) all real numbers (B) $[1,3]$ (C) $(0,4)$ (D) $[0,4]$ (E) $[0,2) \cup (2,4]$

Answer: E We need nonnegative numbers inside each square root, so we need both $x \geq 0$ and $4 - x \geq 0$, thus $x \geq 0$ and $x \leq 4$, meaning $0 \leq x \leq 4$, or in interval notation, $[0,4]$. However, we must also avoid dividing by zero. Thus, we solve $\sqrt{x} - \sqrt{4-x} = 0$ to get $x = 4 - x$, or $x = 2$. Hence, $x = 2$ is *not* part of the domain, resulting in a domain of $[0,2) \cup (2,4]$.

8. Suppose $r = a/b$, with a and b distinct nonzero integers. Which of the following is equivalent to $a/(a-b)$?

(A) $r-1$ (B) $\frac{r+1}{r}$ (C) $\frac{r-1}{r}$ (D) $\frac{r+1}{r-1}$ (E) $\frac{r}{r-1}$

Answer: E Since $r = a/b$, then $a = br$. We have

$$\frac{a}{a-b} = \frac{br}{br-b} = \frac{br}{b(r-1)} = \frac{r}{r-1}.$$

9. Compute the number of zeroes at the end of 105!.

(A) 21 (B) 23 (C) 24 (D) 25 (E) 27

Answer: D The number of zeroes can be calculated by counting the number of factors of 5 in 105!. (This is because a zero on the end comes from a factor of $10 = 2 \times 5$, and since there are far more factors of 2 than factors of 5 in a factorial, each factor of 5 will be combined with a 2 to produce a 10, which creates a zero on the end. So we count the factors of 5.) Since 105 divided by 5 is 21, there are at least 21 zeros. However, there are extra factors of 5 in each power of 5 (25, 50, 75, 100) which contribute another 4 zeroes. Thus the total number of zeroes is 25.

10. Compute the sum of the solutions to the equation $\sin^2\theta - \sin\theta = 1$, where θ is in radians, on the interval $[0, 2\pi)$.

(A) π (B) $\frac{6\pi}{5}$ (C) 3π (D) $\frac{9\pi}{2}$ (E) 5π

Answer: C The equation $\sin^2\theta - \sin\theta - 1 = 0$ is quadratic in $\sin\theta$, so we obtain

$$\sin\theta = \frac{1 \pm \sqrt{5}}{2}.$$

However, taking the positive root gives a value that is larger than 1, so this yields no solutions for θ. Taking the negative root gives a value that is less than 0 and greater than -1, so the solution

$$\theta = \sin^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$$

gives a negative angle in the fourth quadrant, because the range of the inverse sine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus the angle which lies in the interval $[0, 2\pi)$ in the fourth quadrant is $2\pi + \theta$. There is a corresponding angle in the third quadrant as well: $\pi - \theta$. Therefore, the sum of these two solutions is $2\pi + \theta + \pi - \theta = 3\pi$.

11. Evaluate $\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{2016}$.

(A) 1 (B) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ (C) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (D) $1008+1008i\sqrt{3}$ (E) $1008\sqrt{3}-1008i$

Answer: A Convert the complex number to polar. The real part of $-\frac{1}{2}$ and the imaginary part of $\frac{\sqrt{3}}{2}$ implies that the angle θ is $\theta=\frac{5\pi}{6}$. (Note that the magnitude of $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ is 1.) Then we use DeMoivre's Theorem:

$$\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{2016}=\left(e^{5i\pi/6}\right)^{2016}=e^{2016\cdot 5i\pi/6}=e^{1680i\pi}.$$

We have an even multiple of π, so the angle $\theta=1680\pi$ is equivalent to $\theta=0$. Hence, $e^0=1$.

12. Given that $|x|\neq|y|$, simplify $\frac{x^2-xy+y^2}{x^3+y^3}-\frac{x^2+xy+y^2}{x^3-y^3}$.

(A) $2x$ (B) $\frac{1}{x^6-y^6}$ (C) $\frac{-2y}{x^4-x^2y^2+y^4}$ (D) $\frac{2x}{x^2-y^2}$ (E) $\frac{2y}{y^2-x^2}$

Answer: E We have

$$\begin{aligned}\frac{x^2-xy+y^2}{x^3+y^3}-\frac{x^2+xy+y^2}{x^3-y^3}&=\frac{x^2-xy+y^2}{(x+y)\left(x^2-xy+y^2\right)}-\frac{x^2+xy+y^2}{(x-y)\left(x^2+xy+y^2\right)}\\&=\frac{1}{x+y}-\frac{1}{x-y}\\&=\frac{x-y-(x+y)}{(x+y)(x-y)}\\&=\frac{-2y}{x^2-y^2}\\&=\frac{2y}{y^2-x^2}.\end{aligned}$$

13. The polynomial $P(x)=2x^4-3x^3+ax+b$, where a and b are real numbers, can be divided evenly by x^2-4. Compute $a+b$.

(A) -20 (B) -12 (C) 8 (D) 12 (E) 20

Answer: A Since $x^2-4=(x-2)(x+2)$, this implies that $P(x)$ can be divided by both $x-2$ and $x+2$. By the Remainder Theorem, we have that $P(2)=0$ and $P(-2)=0$. This gives the system of equations

$$\begin{cases}32-24+2a+b=0\\32+24-2a+b=0\end{cases}\quad\text{which simplifies to}\quad\begin{cases}2a+b=-8\\2a-b=\ \ 56.\end{cases}$$

The solution to the system is $a=12$, $b=-32$, and their sum is $a+b=12-32=-20$.

14. Compute the units digit of the sum of the coefficients of the expansion of $(8+x)^{2016}$.

(A) 1 (B) 2 (C) 6 (D) 8 (E) 9

Answer: A The sum of the coefficients is given by letting $x = 1$. This yields 9^{2016}. So we are looking for the units digit of 9^{2016}. It is easily established that even powers of 9 have a units digit of 1 (and odd powers of 9 have a units digit of 9).

15. Compute the sum of all possible values of k so that the piecewise function f, defined below, is continuous.

$$f(x) = \begin{cases} (k+1)x + 10 & \text{for } x \le k \\ 2x^2 + 3k - 5 & \text{for } x > k \end{cases}$$

(A) -6 (B) -2 (C) 1 (D) 2 (E) 6

Answer: B We need the y-values of each piece to approach to the same value as x approaches k from both sides. Therefore, $(k+1)x + 10$ must equal $2x^2 + 3k - 5$ when $x = k$. So we have

$$\begin{aligned} (k+1)k + 10 &= 2k^2 + 3k - 5 \\ k^2 + k + 10 &= 2k^2 + 3k - 5 \\ k^2 + 2k - 15 &= 0 \\ (k+5)(k-3) &= 0. \end{aligned}$$

So, $k = -5$ or $k = 3$, and their sum is -2.

16. What is the probability of rolling a fair die four times and observing the 5 face up at least twice?

(A) $\frac{5}{324}$ (B) $\frac{25}{1296}$ (C) $\frac{19}{324}$ (D) $\frac{19}{144}$ (E) $\frac{125}{324}$

Answer: D We use complementary probability to compute this: the probability of at least two 5s is equal to the probability of no 5s plus exactly one 5, subtracted from 1. Hence, the probability is

$$1 - \left(\frac{5}{6}\right)^4 - \binom{4}{3}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = 1 - \frac{625}{1296} - \frac{500}{1296} = \frac{171}{1296} = \frac{19}{144}.$$

17. The counting numbers are arranged in rows with one number in the first row, two numbers in the second row, three numbers in the third row, and so forth, as shown below.

		columns					
		1	2	3	4	5	...
	1	1					
	2	2	3				
rows	3	4	5	6			
	4	7	8	9	10		
	5	11	12	13	14	15	
	⋮						⋱

What is the sum of the row number and column number for the entry 2016? (For example, in the diagram above, the entry 9 is in row 4 column 3 so the sum is 7.)

(A) 63 (B) 64 (C) 65 (D) 125 (E) 126

Answer: E Note that the numbers on the end of each row are the triangular numbers 1, 3, 6, 10, 15, etc. The triangular numbers are represented by $n(n+1)/2$. Thus we find n so that

$$\frac{n(n+1)}{2} \leq 2016 < \frac{(n+1)(n+2)}{2}$$

or $n(n+1) \leq 4032 < (n+1)(n+2)$. Since $4032 = 63 \cdot 64$, we see that 2016 is the 63rd triangular number. This means that 2016 is in the 63rd row and the 63rd column, giving a sum of 126.

18. Three marbles are to be drawn from among three urns. The first will be randomly drawn from an urn that contains exactly 8 blue marbles and 2 yellow marbles. The second will be randomly drawn from an urn that contains exactly 8 red marbles and 2 yellow marbles. The third will be randomly drawn from an urn that contains exactly 8 green marbles and 2 yellow marbles. To the nearest hundredth, what is the probability that at least one yellow marble will be drawn?

(A) 0.33 (B) 0.49 (C) 0.50 (D) 0.51 (E) 0.67

Answer: B Since the probability that at least one yellow marble is drawn plus the probability that no yellow marbles are drawn is equal to 1, we compute the probability that no yellow marbles are drawn and subtract that from 1. Thus,

$$P(\text{no yellow marbles}) = \frac{8}{10} \cdot \frac{8}{10} \cdot \frac{8}{10} = \frac{512}{1000} = 0.512.$$

Finally, the probability that at least one yellow marble is drawn is $1 - 0.512 = 0.488$. Rounded to the nearest hundredth, this is 0.49.

19. Dr. Beaker made a mistake preparing the HCl solution for this week's lab. Instead of preparing 5 liters of 2 M HCl, she prepared 5 liters of 1 M HCl. (A 1 M solution of hydrochloric acid contains 1 mole of HCl per liter of solution.) If she uses all of the 1 M HCl solution she has prepared, how many liters of 12 M stock solution should she add to get a 2 M HCl solution?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 1 (E) 2

Answer: B She has 5 liters of 1 M HCl. This solution contains (5 L)(1 mole/L) = 5 moles HCl. She is adding x liters of 12 M HCl solution which contains $12x$ moles HCl. We can calculate the concentration by dividing the number of moles by the volume. This yields

$$\begin{aligned}\frac{(5+12x)\text{ mol}}{(5+x)\text{ L}} &= \frac{2\text{ mol}}{\text{L}}\\ 5+12x &= 10+2x\\ 10x &= 5\\ x &= \frac{1}{2}.\end{aligned}$$

20. Suppose $\sin A + \cos B = 1.5$ and $\sin A - \cos B = 0.3$, where A and B are both first quadrant angles. The exact value of $\sin(A+B)$ can be written as $\frac{m+n\sqrt{p}}{q}$, where m, n, p, and q are positive integers, m and q are relatively prime, and p is a prime number. Compute $m+n+p+q$.

(A) 58 (B) 86 (C) 100 (D) 176 (E) None of these

Answer: C Note that adding the given equations gives us $2\sin A = 1.8$, or $\sin A = 0.9$. Subtracting the given equations yields $2\cos B = 1.2$, or $\cos B = 0.6$. Then

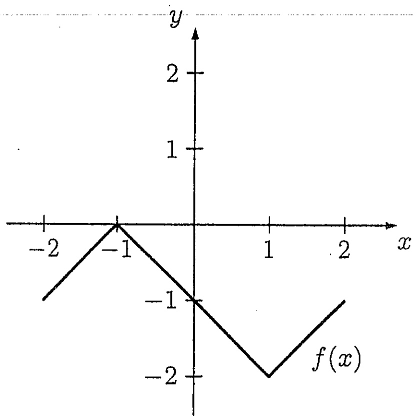
$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-0.81} = \sqrt{0.19}$ and $\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-0.36} = \sqrt{0.64}$.

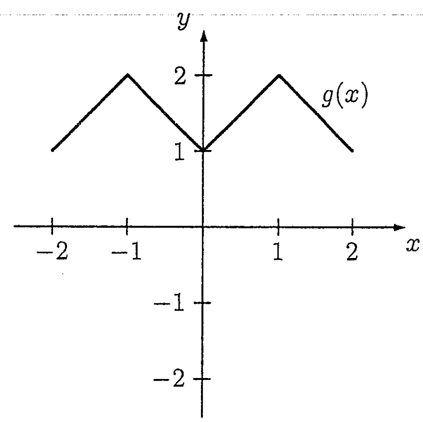
We neglected the negative roots since the angles are in the first quadrant. Next, we use the angle addition identity for sine:

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \sin B\cos A\\ &= 0.9\cdot 0.6 + \sqrt{0.19\cdot 0.64}\\ &= 0.54 + 0.08\sqrt{19}\\ &= \frac{54+8\sqrt{19}}{100}\\ &= \frac{27+4\sqrt{19}}{50}.\end{aligned}$$

Finally, we have $m+n+p+q = 27+4+19+50 = 100$.

21. The graphs of functions f and g, for $-2 \leq x \leq 2$, are shown below.





Which of the following statements is/are true?

I. $g(x) = -f(|x|)$
II. $g(x) = -|f(x)|$
III. $g(x) = |f(|x|)|$

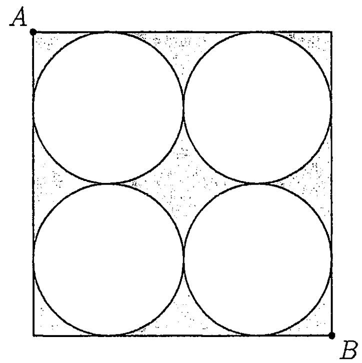
(A) I only (B) II only (C) III only (D) I and II only (E) I and III only

Answer: E When we replace $f(x)$ with $f(|x|)$, the portion of the graph to the right of the y-axis will not change, and it will also be reflected across the y-axis, as the new function is even. When we replace $f(x)$ with $|f(x)|$, the portion of the graph above the x-axis will not change, and any portion below is replaced by its reflection across the x-axis, as all new y-values are positive. Also, when we replace y with $-y$, we reflect the entire graph over the x-axis.

Note that the graph of $f(x)$ is (except where $x = -1$) below the x-axis. Hence, applying $|f(x)|$ reflects the entire graph above the x-axis (except where $x = -1$), and then applying the negative—that is, $-|f(x)|$—reflects the graph right back where it was. Thus, $-|f(x)| = f(x)$. So II is not correct.

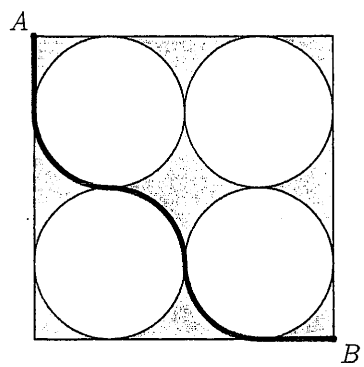
However, applying the transformation $f(|x|)$ results in a "W"-shaped graph, which when reflected over the x-axis, gives us the graph of g. Since both the negative and the absolute value transformation result in this reflection, both I and III are correct.

22. In the figure below, four congruent circles each of radius 1 unit are packed into a square. If a particle must travel a path within the shaded area or along its boundary, compute the length of the shortest path from A to B.



(A) $2 + \frac{3\pi}{2}$ (B) $4 + \pi$ (C) $4\left(\sqrt{2} - 1\right) + 2\pi$ (D) 8 (E) $4 + 4\sqrt{2}$

Answer: A The shortest path is shown below.

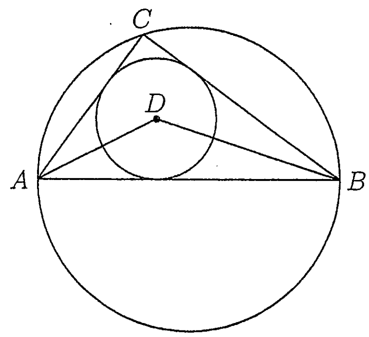


This is equivalent to the lengths of two radii plus $\frac{3}{4}$ the circumference of one of the circles; hence, the path has length $1 + 1 + \frac{3}{4}(2\pi \cdot 1) = 2 + \frac{3\pi}{2}$.

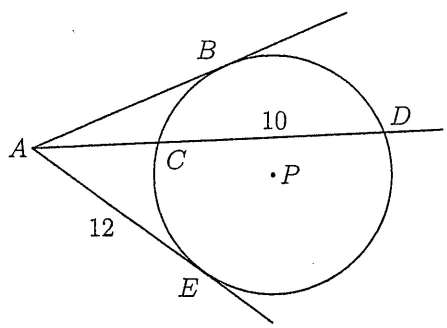
23. A circle has diameter $\overline{AB}$ with A and B on the circle. Point C is on the circle, but at neither A nor B. A circle with center D is inscribed in $\triangle ABC$. Compute the measure of $\angle ADB$ in degrees.

(A) 90° **(B)** 105° **(C)** 135° **(D)** 150° **(E)** Cannot be determined

Answer: C Since $\triangle ABC$ has its vertices on the circle and $\overline{AB}$ is a diameter, then $\angle ACB$ is right. Also, point D is the incenter of $\triangle ABC$, which implies $\overline{DA}$ and $\overline{DB}$ are angle bisectors. Since $\angle ACB$ is right, the other two angles in $\triangle ABC$ sum to 90°. Thus, the angles formed by the angle bisectors (namely, $\angle DAB$ and $\angle DBA$) must sum to half of 90°, or 45°. Thus, in $\triangle ADB$, we have $\angle DAB + \angle DBA + \angle ADB = 45° + \angle ADB = 180°$, or $\angle ADB = 135°$.



24. In the diagram below, $\overline{AE}$ and $\overline{AB}$ are tangent to circle P, and a line from A intersects circle P at C and D. Given that $AE = 12$ and $CD = 10$, compute $AC \cdot AB$.



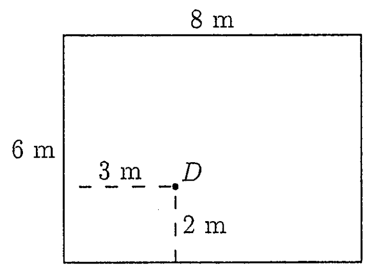
(A) 24 **(B)** 48 **(C)** 72 **(D)** 96 **(E)** Cannot be determined

Answer: D Tangent segments from the same point are congruent, so $AE = 12 = AB$. Furthermore, $\triangle AEC \sim \triangle ADE$, so

$$\frac{12}{AC+10} = \frac{AC}{12}$$
$$12 \cdot 12 = AC(AC+10)$$
$$144 = AC^2 + 10AC$$
$$AC^2 + 10AC - 144 = 0$$
$$(AC+18)(AC-8) = 0,$$

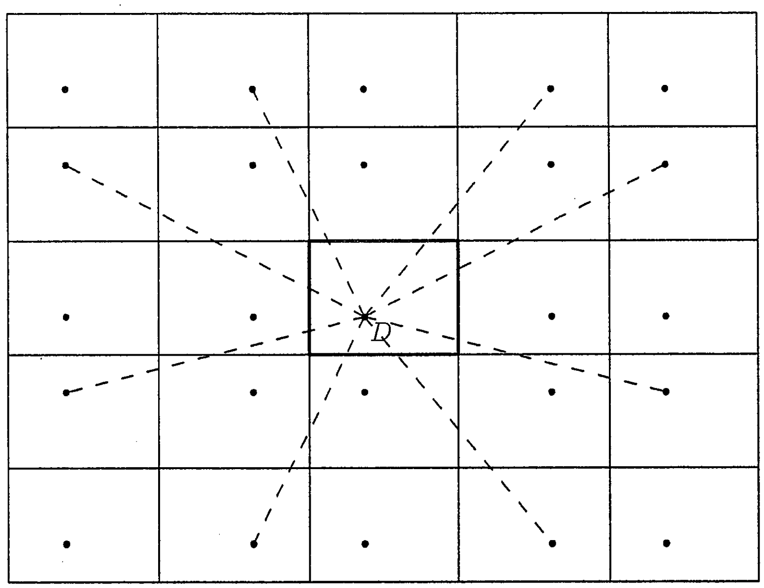
which implies $AC = 8$, since AC cannot be negative. Therefore $AC \cdot AB = 8 \cdot 12 = 96$.

25. An evil mutated alien Kaled from the planet Skaro is trapped! The alien is trapped in a room which is impervious to laser-beam weapons! The rectangle below represents the room with dimensions 6×8 meters. The Kaled is positioned 2 meters from the 8-meter wall and 3 meters from the 6-meter wall. It fires a laser beam weapon from point D. The laser beam reflects off of three walls (avoiding corners) before returning to point D and exterminating itself. What is the minimum length traversed by the beam?



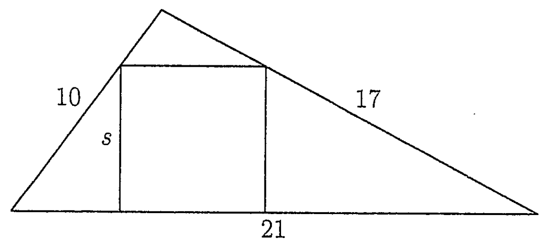
(A) $6\sqrt{5}$ (B) $4\sqrt{13}$ (C) $2\sqrt{61}$ (D) $4\sqrt{20}$ (E) $5\sqrt{65}$

Answer: A FIRST SOLUTION. Rather than deal with reflections, imagine the various mirror images of the room and laser beam were real, existing on the other sides of the wall, as shown below. (*Note:* the middle room is the "real" one, the rest correspond to the various reflections)



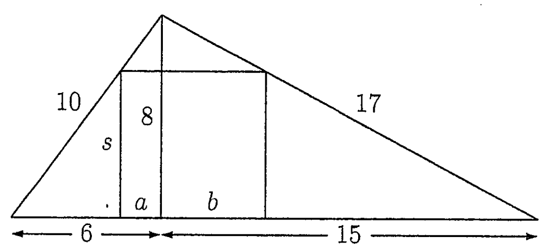
In this way, the shortest reflected path that starts and ends at D can be represented as straight path from D to one of the "reflections of D". (The number of times this straight path crosses one of the lines in the figure shown above equals the number of times the laser beam is reflected off of a wall.) Recalling that the laser beam reflects off of three walls before returning to D and avoids corners, and after accounting for symmetry, there are really only four paths to consider. These are shown as dashed lines in the figure. The shortest of these segments gives the minimum length traversed by the laser beam. We can use the distances between D and the walls along with the Pythagorean Theorem to calculate the length of each dashed line. The shortest is the second one from the left, with length $\sqrt{6^2 + 12^2} = 6\sqrt{5}$.

26. A square is inscribed in a triangle with side lengths 10, 17, and 21, such that two vertices of the square are on the long side of triangle and one vertex is on each of the other sides, as shown below. What is the side length of the square, rounded to the nearest hundredth?



(A) 5.28 **(B)** 5.79 **(C)** 5.93 **(D)** 6.21 **(E)** 6.36

Answer: B Dropping an altitude to the side of length 21 splits this triangle into two right triangles: the 6-8-10 triangle and the 8-15-17 triangle, where the altitude is length 8.



In the figure, the altitude divides the bottom side of the square into lengths a and b. Using similar triangles, we have

$$\frac{a}{6} = \frac{8-s}{8} \qquad \text{or} \qquad a = 6 - \frac{6}{8}s,$$

and

$$\frac{b}{15} = \frac{8-s}{8} \qquad \text{or} \qquad b = 15 - \frac{15}{8}s.$$

Also, since $a + b = s$, then

$$s = a + b = 6 - \frac{6}{8}s + 15 - \frac{15}{8}s = 21 - \frac{21}{8}s.$$

Solving for s, we get $s = \frac{168}{29} \approx 5.79$.

27. Your little brother rips a bunch of consecutive pages out of your favorite mathematics textbook. Given that the pages are printed back-to-back, and the sum of the page numbers which were ripped out is 1616, what was the number of the first page ripped out?

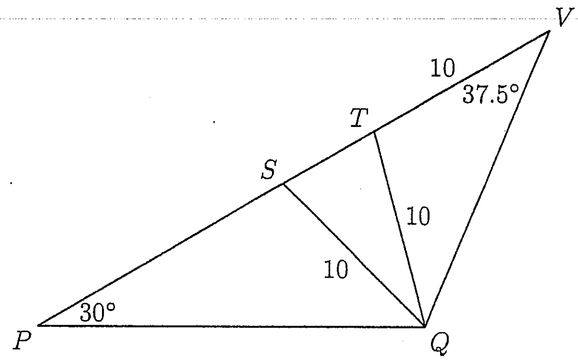
(A) 16 (B) 35 (C) 66 (D) 68 (E) 101

Answer: B Let k be the number of the last page ripped out, and let n be the number of first page ripped out. Then the sum of the page numbers from 1 to k is $\frac{1}{2}k(k+1)$ and the sum of the page numbers from 1 to $n-1$ is $\frac{1}{2}(n-1)(n)$. The difference in these two sums must be 1616. Hence, we have

$$\begin{aligned}\frac{k(k+1)}{2}-\frac{n(n-1)}{2}&=1616\\ k^2+k-n^2+n&=3232\\ k^2-n^2+k+n&=3232\\ (k+n)(k-n)+(k+n)&=3232\\ (k+n)(k-n+1)&=3232.\end{aligned}$$

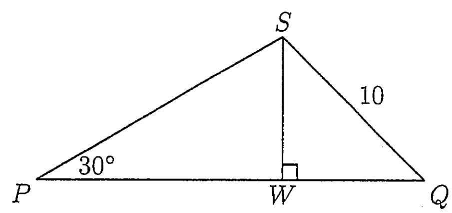
Since 3232 factors into $32\cdot 101$, we have that $k+n=101$ and $k-n+1=32$. Solving this sytem of equations gives us $k=66$ and $n=35$. Thus, the first page ripped out was 35.

28. In the figure below, $\angle P = 30°$, $\angle V = 37.5°$, and $SQ = TQ = TV = 10$. Compute PQ.



(A) $5\sqrt{3}+5$ **(B)** $10\sqrt{2}$ **(C)** $10\sqrt{3}$ **(D)** $5\sqrt{2}+5\sqrt{6}$ **(E)** $20\sqrt{2}$

Answer: D Note that $\triangle QTV$ is isosceles with base angles 37.5°; hence $\angle QTV = 105°$. From this, we see that $\angle QTS = 180° - 105° = 75°$, and since $\triangle TQS$ is also isosceles, we have $\angle QST = 75°$. This implies $\angle QSP = 105°$. This, with the fact that $\angle P = 30°$, gives $\angle SQP = 45°$. Now we focus on $\triangle QSP$.

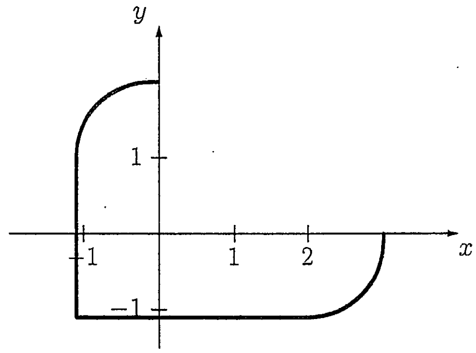


FIRST SOLUTION. We drop an altitude from S to $\overline{PQ}$ at W which creates a 30°-60°-90° triangle and a 45°-45°-90° triangle. Since $SQ = 10$, we have $WQ = 10/\sqrt{2} = 5\sqrt{2}$, so that $WS = 5\sqrt{2}$ also. Then $PW = 5\sqrt{2}\cdot\sqrt{3} = 5\sqrt{6}$. Finally, $PQ = PW + WQ = 5\sqrt{2} + 5\sqrt{6}$.

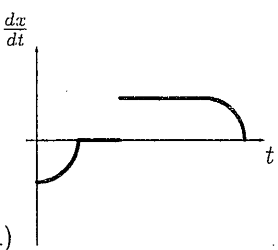
SECOND SOLUTION. Starting with $\triangle QSP$, we use the Law of Sines and the angle addition indentity for sine.. We have

$$\begin{aligned}
\frac{\sin\angle P}{SQ} &= \frac{\sin\angle S}{PQ} \\
\frac{\sin 30°}{10} &= \frac{\sin 105°}{PQ} \\
PQ &= \frac{10\sin 105°}{\sin 30°} = \frac{10\sin(60° + 45°)}{1/2} \\
&= 20\left(\sin 60°\cos 45° + \sin 45°\cos 60°\right) \\
&= 20\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right) \\
&= 20\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) \\
&= 5\sqrt{6} + 5\sqrt{2}.
\end{aligned}$$

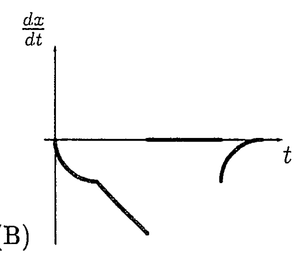
29. The graph below describes the path of a particle. The particle starts at $(0, 2)$ and moves along the path shown at a constant speed, ending at $(3, 0)$.



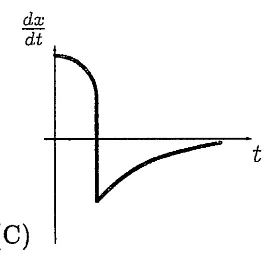
Which graph best describes $\dfrac{dx}{dt}$ for the particle during this journey?



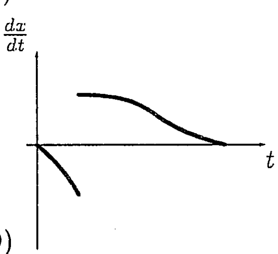
(A)



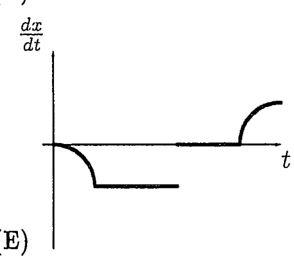
(B)



(C)



(D)



(E)

Answer: A The horizontal velocity of the particle is represented by dx/dt, the rate of change of the x-coordinate with repsect to time. The original graph implies that the particle starts backwards (implying $dx/dt < 0$), then moves only vertically (so $dx/dt = 0$), then moves steadily to the right (so $dx/dt > 0$ and dx/dt is constant), then slows down horizontally. Only the graph in choice A matches the requirements of this horizontal movement.

30. In space, a vector from point B to point A is given by $\langle 1, -2, 3\rangle$, and a vector from point B to point C is given by $\langle 2, 1, -3\rangle$. Compute $\sin \angle ABC$.

(A) 0 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{115}}{14}$ (D) $\frac{\sqrt{3}}{2}$ (E) 1

Answer: C The cosine of the angle between two vectors is the dot product of the vectors divided by the product of their magnitudes. We have

$$\cos \angle ABC = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\|\overrightarrow{BC}\|} = \frac{1 \cdot 2 + (-2)1 + 3(-3)}{\sqrt{1 + (-2)^2 + 3^2}\sqrt{2^2 + 1 + (-3)^2}} = \frac{-9}{\sqrt{14}\sqrt{14}} = -\frac{9}{14}.$$

Then

$$\sin \angle ABC = \sqrt{1 - \cos^2 \angle ABC} = \sqrt{1 - \left(-\frac{9}{14}\right)^2} = \sqrt{1 - \frac{81}{196}} = \sqrt{\frac{115}{196}} = \frac{\sqrt{115}}{14}.$$

31. Let k be the smallest natural number such that $k! + (k-1)!$ is divisible by 256. Compute the remainder when k is divided by 5.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Answer: E We note that $256 = 2^8$, so we require at least eight factors of 2 in $k! + (k-1)! = k(k-1)! + (k-1)! = (k-1)!(k+1)$. It is each even value of $k-1$ (in other words, each odd value of k) which contributes another power of 2 in $(k-1)!$. And of course, if k is odd, then $k+1$ is even, so that contributes more powers of 2. Noting that 8! contains seven factors of 2 and 10! contains eight factors of 2, we choose as our starting point $k = 9$. Then

$$9! + (9-1)! = (9-1)!(9+1) = 8!10 = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 3 \cdot 2 \cdot 7 \cdot 2^3 \cdot 2 \cdot 5,$$

which includes exactly eight powers of 2, so this is indeed divisible by 256. However, to be sure, we check $k = 7$ (the next smaller odd number). Then

$$7! + (7-1)! = (7-1)!(7+1) = 6!8 = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot 3 \cdot 2 \cdot 2^3.$$

This includes exactly seven powers of 2, so this is not divisible by 256, and $k = 9$ must be the correct value of k. The remainder when 9 is divided by 5 is 4.

32. The sum

$$\sum_{n=1}^{\infty} \frac{\pi n - 1}{4^n} = \frac{\pi - 1}{4} + \frac{2\pi - 1}{16} + \frac{3\pi - 1}{64} + \frac{4\pi - 1}{256} + \cdots$$

can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are integers and a and c are relatively prime. Compute $a + b + c$.

(A) 16 (B) 19 (C) 20 (D) 22 (E) 25

Answer: A Call the sum S. Then take $\frac{1}{4}S$. We have the two equations

$$S = \frac{\pi - 1}{4} + \frac{2\pi - 1}{16} + \frac{3\pi - 1}{64} + \frac{4\pi - 1}{256} + \cdots$$
$$\frac{1}{4}S = \frac{\pi - 1}{16} + \frac{2\pi - 1}{64} + \frac{3\pi - 1}{256} + \frac{4\pi - 1}{1024} + \cdots.$$

Now subtract these equations to get

$$S - \frac{1}{4}S = \frac{3}{4}S = \frac{\pi - 1}{4} + \frac{\pi}{16} + \frac{\pi}{64} + \frac{\pi}{256} + \cdots.$$

This series (after the term $\frac{\pi - 1}{4}$) is geometric with first term $\frac{\pi}{16}$ and common ratio $\frac{1}{4}$. Hence,

$$\frac{3}{4}S = \frac{\pi - 1}{4} + \frac{\frac{\pi}{16}}{1 - \frac{1}{4}} = \frac{\pi - 1}{4} + \frac{\pi}{16} \cdot \frac{4}{3} = \frac{\pi - 1}{4} + \frac{\pi}{12} = \frac{4\pi - 3}{12}$$
$$S = \frac{4}{3} \cdot \frac{4\pi - 3}{12} = \frac{4\pi - 3}{9}.$$

Thus $a = 4$, $b = 3$, and $c = 9$ so that $a + b + c = 16$.

33. Josh and Marcy are playing a dice rolling game. They each roll one fair six-sided die. Whoever rolls the higher number wins. If they roll the same number, there is no winner. If Josh wins, what is the probability that he rolled a 5 or a 6?

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{5}{12}$ (E) $\frac{3}{5}$

Answer: E The probability of each roll is $\frac{1}{6}$. If Josh rolls a 6, his probability of winning is $\frac{5}{6}$ and if Josh rolls a 5, his probability of winning is $\frac{4}{6}$ (recall that ties do not win). Josh's probability of rolling a 5 or 6 and winning is

$$\frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{4}{6} = \frac{9}{36} = \frac{1}{4}.$$

The probability of Josh and Marcy tying is $6 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{6}$. They have equal chances of winning, so the overall chance of Josh winning is $\frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}$. Therefore, the probability that Josh rolled a 5 or 6 if he won is

$$P(5 \text{ or } 6 \mid \text{win}) = \frac{P((5 \text{ or } 6) \cap \text{win})}{P(\text{win})} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{12}{20} = \frac{3}{5}.$$

34. An arithmetic sequence and a geometric sequence have the same first and fourth terms. If the second term of the arithmetic sequence is 215 and its fifth term is 500, what is the third term of the geometric sequence?

(A) 120 (B) 240 (C) 270 (D) 310 (E) 405

Answer: C Call the arithmetic sequence $\{a_n\}$ so that $a_2 = 215$ and $a_5 = 500$, and let the geometric sequence be $\{g_n\}$. Then we find the common difference d of the arithmetic sequence by solving $a_5 = a_2 + 3d$. Thus, $d = (500 - 215)/3 = 95$. Now we can find a_1 and a_4, which are the corresponding terms of the geometric series, g_1 and g_4. Hence, $g_1 = a_1 = a_2 - d = 215 - 95 = 120$, and $g_4 = a_4 = a_5 - d = 500 - 95 = 405$. Using the properties of geometric sequences with common ratio r, we know that $g_4 = g_1 r^3$. Therefore, $r^3 = \frac{g_4}{g_1} = \frac{405}{120} = \frac{27}{8}$, and finally, $r = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$. Solving for the third term, we get $g_3 = g_1 r^2 = 120\left(\frac{3}{2}\right)^2 = 270$.

35. Compute the area of the largest triangle, rounded to the nearest tenth of a square inch, that will fit in a box measuring 6 inches by 8 inches by 3 inches.

(A) 24 (B) 25.6 (C) 26.8 (D) 28.3 (E) 29.2

Answer: D Place the box in coordinate space so that one vertex is at the origin and the other vertices are at $(0,0,3)$, $(0,8,3)$, $(0,8,0)$, $(6,0,0)$, $(6,0,3)$, $(6,8,3)$, $(6,8,0)$. To get the largest triangle, we want the three vertices which are furthest apart. To get the vertices which are furthest apart, choose coodinates in which each pair differs in two values. One choice is $(0,0,0)$, $(0,8,3)$, and $(6,8,0)$. The area is most easily found by computing the magnitude of the cross product of the vectors from the origin to the other two points. The vectors are $\mathbf{u} = \langle 0,8,3\rangle$ and $\mathbf{v} = \langle 6,8,0\rangle$, so that the cross product is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 8 & 3 \\ 6 & 8 & 0 \end{vmatrix} = -24\mathbf{i} + 18\mathbf{j} - 48\mathbf{k}.$$

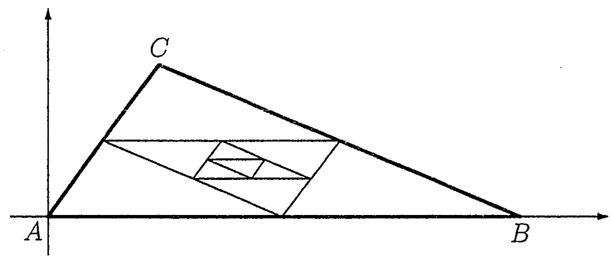
Thus the area A is

$$\begin{aligned} A &= \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\sqrt{(-24)^2 + 18^2 + (-48)^2} \\ &= \frac{1}{2}\sqrt{576 + 324 + 2304} = \frac{1}{2}\sqrt{3204} = 3\sqrt{89}, \end{aligned}$$

Which is approximately 28.3.

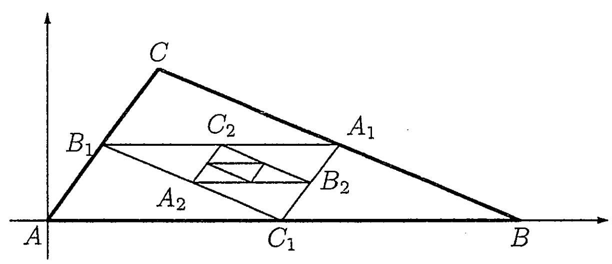
Note: Some of the other answer choices are areas of other triangles formed by using other vertices of the box. Using $(0,0,0)$, $(6,0,0)$, and $(0,8,0)$ gives an area of 24; using $(0,0,0)$, $(0,8,3)$, and $(6,8,3)$ gives an area of 25.6; using $(0,0,0)$, $(0,8,0)$, and $(6,8,3)$ gives an area of 26.8.

36. Let $\triangle ABC$ have vertices $A(0,0)$, $B(20,0)$, and $C(4,18)$. As shown below, the midpoints of the sides are connected to form a smaller inner triangle. Then, the midpoints of the sides of this smaller triangle are connected in a similar way to form an even smaller triangle. This process is continued *ad infinitum*. Let (x,y) be the coordinates of the only point contained in all such constructed inner triangles. Compute $x+y$.



(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Answer: D Let's label the various vertices of the triangles drawn in the following systematic way, so that the first inner constructed triangle is $\triangle A_1B_1C_1$, the second is $\triangle A_2B_2C_2$, the third is $\triangle A_3B_3C_3$, and so on.



Note that B_2 is the midpoint of diagonal $\overline{A_1C_1}$ in parallelogram $B_1A_1BC_1$ and thus must lie on the other diagonal of that parallelogram, namely $\overline{BB_1}$. So B, B_1, and B_2 are collinear.

In a similar way, we can argue B_1, B_2, and B_3 are collinear. Indeed, by considering the $\triangle A_iB_iC_i$, we can argue that B_i, B_{i+1} and B_{i+2} are collinear. Given the pairwise overlap of these sets of points, all of the points B_i, for $i = 1,2,3,\ldots$ must then be collinear. For similar reasons, we can see that all of the points A_i, for $i = 1,2,3,\ldots$ must be collinear, and all of the points C_i, for $i = 1,2,3,\ldots$ must be collinear.

Given that A_1, B_1, and C_1 are midpoints of their respective sides, we can see that all of aforementioned points B_i lie on the median through B, all of the aforementioned points A_i lie on the median through A, and all of the aforementioned points C_i lie on the median through C.

Consequently, all three medians must intersect at a common point—one contained in all of the inner triangles constructed (the very point we seek!) Only two of these medians are required to find this common intersection point, so we find the medians through A and B.

The median through A must go through $(0,0)$ and $(12,9)$, which yields $y = \frac{3}{4}x$.

The median through B must go through $(2,9)$ and $(20,0)$, yielding $y = -\frac{1}{2}x + 10$.

The intersection of these two lines is $(8,6)$, so $x+y = 14$.

37. The function $f(\theta)$ is defined as the following infinite expression.

$$f(\theta) = -\sin(-\theta) + \cos^2(90^\circ - \theta) - \frac{1}{\csc^3(-\theta)} + \frac{1}{\sec^4(90^\circ - \theta)} - \sin^5(-\theta) + \cos^6(90^\circ - \theta) - \frac{1}{\csc^7(-\theta)} + \frac{1}{\sec^8(90^\circ - \theta)} + \cdots$$

Compute $f(60^\circ)$.

(A) 0 **(B)** 1 **(C)** $3\sqrt{2}+2$ **(D)** $2\sqrt{3}+3$ **(E)** undefined

Answer: D Note that

$$-\sin(-\theta) = \sin(\theta), \qquad \cos(90^\circ - \theta) = \sin(\theta), \qquad \frac{-1}{\csc(-\theta)} = \sin(\theta), \qquad \frac{1}{\sec(90^\circ - \theta)} = \sin(\theta).$$

Then the function becomes

$$f(\theta) = \sin(\theta) + \sin^2(\theta) + \sin^3(\theta) + \sin^4(\theta) + \sin^5(\theta) + \sin^6(\theta) + \sin^7(\theta) + \cdots.$$

Thus, this function is an infinite geometric series with common ratio $\sin(\theta)$. Since $\sin(60^\circ) = \frac{\sqrt{3}}{2}$, we have $-1 < \sin(60^\circ) < 1$, so the series represented by $f(60^\circ)$ converges. Noting that the first term is $\frac{\sqrt{3}}{2}$ and the common ratio is $\frac{\sqrt{3}}{2}$, then we have the sum

$$f(60^\circ) = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2-\sqrt{3}} = \frac{\sqrt{3}\,(2+\sqrt{3})}{(2-\sqrt{3})\,(2+\sqrt{3})} = \frac{2\sqrt{3}+3}{4-3} = 2\sqrt{3}+3.$$

38. Evaluate the base-2 number 0.110110110110... as a reduced fraction in base-10.

(A) $\frac{2}{3}$ (B) $\frac{4}{5}$ (C) $\frac{6}{7}$ (D) $\frac{8}{9}$ (E) $\frac{10}{11}$

Answer: C FIRST SOLUTION. This base-2 decimal ("binarimal"?) is really two intertwined geometric series:

$$\begin{aligned} 0.110110110110\ldots &= \frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32}+\frac{1}{128}+\frac{1}{256}+\cdots \\ &= \left(\frac{1}{2}+\frac{1}{16}+\frac{1}{128}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{32}+\frac{1}{256}+\cdots\right) \\ &= \frac{\frac{1}{2}}{1-\frac{1}{8}}+\frac{\frac{1}{4}}{1-\frac{1}{8}} \\ &= \frac{4}{7}+\frac{2}{7}. \end{aligned}$$

Thus, $0.110110110110\ldots = \frac{4}{7}+\frac{2}{7}=\frac{6}{7}$.

SECOND SOLUTION, SIMILAR TO THE FIRST. Note that, in base-2,

$$0.110110110110\ldots = 0.1111111111\ldots - 0.001001001001\ldots.$$

Now,

$$0.1111111111\ldots = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1,$$

since this is an infinite geometric series. Likewise,

$$0.001001001001\ldots = \frac{1}{8}+\frac{1}{64}+\frac{1}{512}+\cdots = \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7}.$$

Thus, $0.110110110110\ldots = 1-\frac{1}{7}=\frac{6}{7}$.

THIRD (SLICK) SOLUTION. Let $x = 0.\overline{110}_2$. Then multiply both sides by $8 = 2^3$ to get $8x = 110.\overline{110}_2$. Now subtract the equation $x = 0.\overline{110}_2$ from the equation $8x = 110.\overline{110}_2$ to get $7x = 110_2$. In base 2, we have that $110_2 = 6_{10}$. Then $7x = 6$ and so $x = \frac{6}{7}$.

39. The letters of the word `AMBIDEXTROUSLY` undergo a particular permutation repeatedly. Applying the permutation once results in the word `ABMDEITROUXLYS`. Applying it again results in `AMBEIDROUXTYSL`. We apply this permutation again and again and again *ad infinitum*. Let P be the minimum number of permutations which return the letters to their original order. What is the sum of the digits of P?

(A) 3 (B) 5 (C) 7 (D) 9 (E) 11

Answer: A The permutation acts on the letters in the following fashion.

- A → A
- B → M → B
- D → E → I → D
- T → R → O → U → X → T
- L → Y → S → L

Thus,

- The letter A is always in its original position
- The letters B and M are in their original positions every 2 permutations
- The letters D, E, I, and L, Y, S are in their original positions every 3 permutations
- The letters T, R, O, U, and X are in their original positions every 5 permutations

So all of the letters will be in their original position after P scramblings where P is a multiple of 2, 3, and 5. The smallest such P is the product of these numbers, 30. Thus, the sum of the digits of P is 3.

40. Mary is at her favorite doughnut shop, Toroidal Treats, trying to decide how many of each type to buy. Her choices are glazed, chocolate, cream, and strawberry doughnuts. If she buys one dozen doughnuts, how many possible choices are there? (She may buy any number of each variety from 0 to 12, and each doughnut of the same variety is assumed to be identical.)

(A) 165 (B) 220 (C) 455 (D) 495 (E) 1365

Answer: C There are 12 doughnuts which need to be divided into four groups, corresponding to the four flavors. This is a perfect situation for the "stars-and-bars" method: we can think of the twelve doughnuts as being the "stars" and the flavors being represented by three "bars" which divide the doughnuts into the four categories of flavors. Thus our situation is equivalent to the number of ways of arranging three bars among twelve stars. Thus the number of combinations is

$$\binom{12+3}{3} = \binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2} = 5 \cdot 7 \cdot 13 = 455.$$

41. Suppose we say that a real number x is *squishy* whenever $x = a + b\sqrt{10}$ for some integers a and b. Additionally, let us classify a squishy number u as *twonky* if and only if the quotient x/u is squishy for every squishy number x. If u is twonky and can be represented as $u = c + d\sqrt{10}$ where c and d are integers, which of the following could be a value for c?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Answer: B Since u is twonky, it is also squishy. Now, u is twonky only when, for all integers a and b, we can find integers M and N so that

$$\frac{a + b\sqrt{10}}{c + d\sqrt{10}} = M + N\sqrt{10}$$

But then, note

$$\begin{aligned}\frac{a + b\sqrt{10}}{c + d\sqrt{10}} &= \frac{a + b\sqrt{10}}{c + d\sqrt{10}} \cdot \frac{c - d\sqrt{10}}{c - d\sqrt{10}} \\ &= \frac{ac - 10bd + (bc - ad)\sqrt{10}}{c^2 - 10d^2} \\ &= \frac{ac - 10bd}{c^2 - 10d^2} + \left(\frac{bc - ad}{c^2 - 10d^2}\right)\sqrt{10}.\end{aligned}$$

Thus, since u is twonky, both $\frac{ac-10bd}{c^2-10d^2}$ and $\frac{bc-ad}{c^2-10d^2}$ must be integers for all integers a and b. This can only be possible if $c^2 - 10d^2 = \pm 1$. Solving for d, we see that

$$d = \pm\sqrt{\frac{c^2 \pm 1}{10}}$$

must be true. Recall d must be an integer, and use this to test each of the five values of c given in the answer choices. Of these, only $c = 3$ yields a possible integer value for d (namely, $d = \pm 1$).

42. Suppose n is the smallest positive integer such that $n!$ has 2016 trailing zeroes. Compute the remainder when n is divided by 100.

(A) 65 (B) 70 (C) 75 (D) 80 (E) 85

Answer: C The number of trailing zeroes corresponds to the number of factors of 10 that $n!$ contains. This of course equals the number of pairs of factors, 2 and 5, that are contained in $n!$. Since the number of factors of 2 in $n!$ are always greater than or equal to those of 5, we can say that the number of trailing zeroes equals the number of prime factors of 5 that $n!$ contains.

Note, that we pick up a factor of 5 for every multiple of 5 in $1, 2, 3, \ldots, n$. We also pick up an additional factor of 5 for every multiple of 25. In a like manner, we pick up factors of 5 for every multiple of 125, 625, and 3125. As the next power of 5 is 15625 and this exceeds $2016 \cdot 5 = 10080$ which has well more than 2016 trailing zeroes, we do not have to consider any power of 5 higher than 3125. Thus, the number of factors of 5 present in $n!$ is given by

$$f(n) = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \left\lfloor \frac{n}{625} \right\rfloor + \left\lfloor \frac{n}{3125} \right\rfloor$$

We would like to find n where $f(n) = 2016$. We can approximate this n by solving the related equation:

$$\frac{x}{5} + \frac{x}{25} + \frac{x}{125} + \frac{x}{625} + \frac{x}{3125} = 2016$$

Multiplying by 3125 and solving, we have

$$\begin{aligned} 625x + 125x + 25x + 5x + 1 &= 6300000 \\ 781x &= 6300000 \\ x &\approx 8066.5 \end{aligned}$$

However, x only approximates n, since the remainders of the divisions above were not discarded. That said, x gets us very close as

$$\begin{aligned} f(8066) &= \left\lfloor \frac{8066}{5} \right\rfloor + \left\lfloor \frac{8066}{25} \right\rfloor + \left\lfloor \frac{8066}{125} \right\rfloor + \left\lfloor \frac{8066}{625} \right\rfloor + \left\lfloor \frac{8066}{3125} \right\rfloor \\ &= 1613 + 322 + 64 + 12 + 2 \\ &= 2013 \end{aligned}$$

We are only missing three more factors of 5. Given that the number of trailing zeroes in $n!$ must not decrease as n increases, we simply move forward two multiples of 5 to 8075. (*Note:* We pick up *three* multiples of 5 when we do this, as 8075 is also a multiple of 25.) Thus, $n = 8075$ and the remainder when 8075 is divided by 100 is 75.

43. Compute $1 + x + x^2 + x^3$ where x is the solution to the equation

$$\sqrt{x + \sqrt{x + \sqrt{x + \cdots}}} = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots}}}.$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Answer: D Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$. Then we have $y = \sqrt{x + y}$. Solving for x, we find $x = y^2 - y$. Of course if $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$, then

$$y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots}}}$$

as well. This means that $y = x + \frac{1}{y}$. Solving this for x, we find $x = y - \frac{1}{y}$. Setting these two expressions for x equal to one another and solving for y, we find

$$\begin{aligned} y^2 - y &= y - \frac{1}{y} \\ y^3 - y^2 &= y^2 - 1 \\ y^2(y - 1) &= (y + 1)(y - 1). \end{aligned}$$

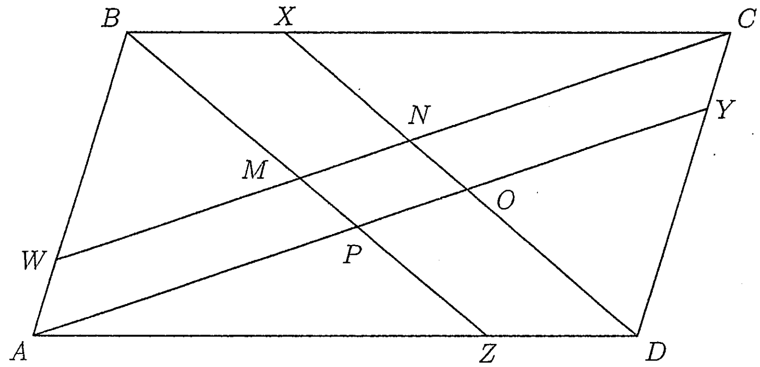
Thus, either $y = 1$ or $y^2 = y + 1$. The last equation can be written $y^2 - y = 1$. Note that if $y = 1$, then $x = 1^2 - 1 = 0$; but this implies $y = 0$ and it cannot be since we assumed $y = 1$. Hence, we reject this solution. Thus, it must be that $x = y^2 - y = 1$. Therefore, $1 + x + x^2 + x^3 = 1 + 1 + 1 + 1 = 4$.

Note: The equation $y^2 - y = 1$ implies $y = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. We can eliminate the negative root since y is positive, so $y = \frac{1}{2}\left(1 + \sqrt{5}\right)$. Then, since $x = y^2 - y$,

$$x = \left(\frac{1 + \sqrt{5}}{2}\right)^2 - \frac{1 + \sqrt{5}}{2} = \frac{6 + 2\sqrt{5}}{4} - \frac{2 + 2\sqrt{5}}{4} = 1$$

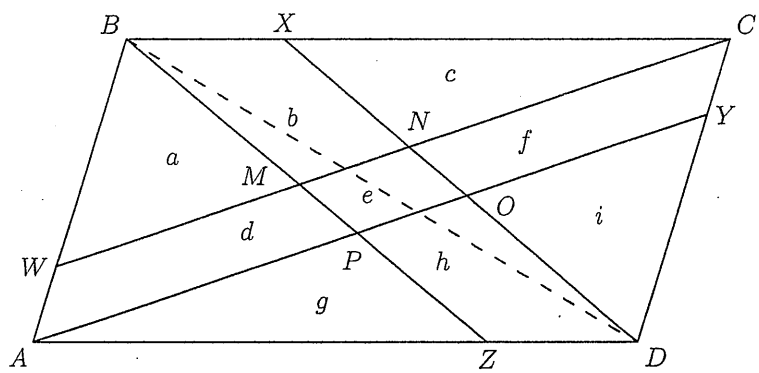
Therefore, $1 + x + x^2 + x^3 = 1 + 1 + 1 + 1 = 4$.

44. In parallelogram $ABCD$, segments AW, BX, CY, and DZ are one fourth as long as segments AB, BC, CD, and DA, respectively. Find the ratio of the area of parallelogram $MNOP$ to the area of parallelogram $ABCD$.



(A) $\frac{1}{21}$ (B) $\frac{1}{22}$ (C) $\frac{1}{23}$ (D) $\frac{1}{24}$ (E) $\frac{1}{25}$

Answer: E FIRST (RIGOROUS) SOLUTION. Let the variables a, b, c, d, e, f, g, h, and i represent the areas of the various triangular and quadrilateral "pieces" of parallelogram $ABCD$, as shown below. Without loss of generality, let us assume the parallelogram has unit area, so that $a+b+c+d+e+f+g+h+i=1$. Draw diagonal BD as shown below. Clearly, the area of $\triangle ABD = \frac{1}{2}$.



Since ZD is one fourth of AD, the area of $\triangle ABZ$ must three-fourths the area of $\triangle ABD$, and thus the area of $\triangle ABZ = \frac{3}{8}$. By similar arguments, the area of $\triangle CDX = \frac{3}{8}$ and the area of $\triangle WBC = \frac{3}{8}$. We will consider $\triangle WBC$ in a moment. However, knowing the areas of $\triangle ABZ$ and $\triangle CDX$ are both $\frac{3}{8}$ tells us the following:

$$\begin{aligned} b+e+h &= 1-(a+d+g)-(c+f+i) \\ &= 1-\frac{3}{8}-\frac{3}{8}=\frac{1}{4} \end{aligned}$$

Note, with b and h tied to clearly congruent areas, it must be true that $b=h$. Thus, $e=\frac{1}{4}-2b$. Observe that $\triangle ABP$ is similar to $\triangle WBM$ and BW is three fourths as long as BA. Also observe that $\triangle BCM$ is similar to $\triangle XCN$ and XC is three fourths as long as BC. Consequently,

$$a=\frac{9}{16}(a+d) \qquad \text{and} \qquad c=\frac{9}{16}(c+b).$$

Solving these last two equations for a and c, respectively, we see

$$a=\frac{9}{7}b \qquad \text{and} \qquad c=\frac{9}{7}b.$$

Earlier we had noticed the area of $\triangle WBC = \frac{3}{8}$. This tells us $a+b+c=\frac{3}{8}$. Substituting for a and c, and solving, we have

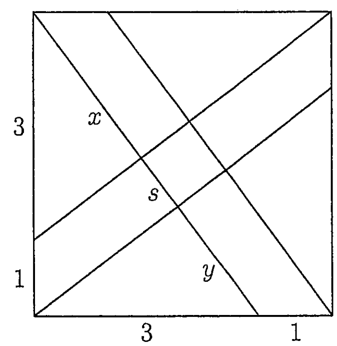
$$\frac{9}{7}b + b + \frac{9}{7}b = \frac{3}{8}$$

so that $b = \frac{21}{200}$. This can then be used to find e:

$$e = \frac{1}{4} - 2b = \frac{1}{4} - \frac{21}{100} = \frac{4}{100} = \frac{1}{25}.$$

Since we assumed parallelogram $ABCD$ had unit area, the ratio of the area of parallelogram $MNOP$ to the area of $ABCD$ is $\frac{1}{25}$.

Second (Not-so rigorous) Solution. The wording of the problem does not specify the shape of the parallelogram, so let us assume it is a square of side length 4. Then we have the diagram below, which has four 3-4-5 right triangles within it.



Using similar triangles, we solve for s, the side length of the center square. We have $\frac{y}{3} = \frac{3}{5}$ so $y = \frac{9}{5}$, and we have $\frac{x}{4} = \frac{3}{5}$ so $x = \frac{12}{5}$. Therefore, $s = 5 - y - x = 5 - \frac{9}{5} - \frac{12}{5} = \frac{4}{5}$. Then the area of the center square is $\frac{16}{25}$ and the area of the large square is 16. The desired ratio is $\frac{1}{25}$.

45. The infinite sum

$$\sum_{n=0}^{\infty} \text{Arctan}\left(\frac{2}{(n+1)^2}\right)$$

can be written in the form $\frac{p}{q}\pi$ where p and q are integers and $\frac{p}{q}$ is a reduced fraction. Compute $p+q$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Answer: D Recall that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Notice that

$$\frac{2}{(n+1)^2} = \frac{(n+2)-n}{1+(n+2)n}$$

takes a similar form. With this in mind, let us find A and B so that $\tan A = n+2$ and $\tan B = n$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \text{Arctan}\left(\frac{2}{(n+1)^2}\right) &= \sum_{n=0}^{\infty} \text{Arctan}\left(\frac{(n+2)-n}{1+(n+2)n}\right)\\ &= \sum_{n=0}^{\infty} \text{Arctan}\left(\frac{\tan A - \tan B}{1+\tan A \tan B}\right)\\ &= \sum_{n=0}^{\infty} \text{Arctan}\left(\tan(A-B)\right)\\ &= \sum_{n=0}^{\infty}(A-B)\\ &= \sum_{n=0}^{\infty}\left(\text{Arctan}\,(n+2) - \text{Arctan}\,(n)\right).\end{aligned}$$

Moreover, we have that $\sum_{n=0}^{m}\left(\text{Arctan}\,(n+2) - \text{Arctan}\,(n)\right)$ telescopes to

$$\text{Arctan}\,(m+2) + \text{Arctan}\,(m+1) - \text{Arctan}\,(1) - \text{Arctan}\,(0).$$

As $m \to \infty$, note Arctan $(m+2)$ and Arctan $(m+1)$ both get arbitrarily close to $\frac{\pi}{2}$. Also, Arctan $(0) = 0$ and Arctan $(1) = \frac{\pi}{4}$ so

$$\sum_{n=0}^{\infty} \text{Arctan}\left(\frac{2}{(n+1)^2}\right) = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} - 0 = \frac{3\pi}{4}.$$

Therefore, $p+q=7$.

46. Simplify $\log_2 4 \cdot \log_4 8 \cdot \log_8 16 \cdot \log_{16} 32 \cdot \log_{32} 64 \cdot \log_{64} 128 \cdot \log_{128} 256 \cdot \log_{256} 512 \cdot \log_{512} 1024$.

Answer: 10 We use the change-of-base formula on each term of the product, which we will call P. This gives

$$\begin{aligned} P &= \log_2 4 \cdot \log_4 8 \cdot \log_8 16 \cdot \log_{16} 32 \cdot \log_{32} 64 \cdot \log_{64} 128 \cdot \log_{128} 256 \cdot \log_{256} 512 \cdot \log_{512} 1024 \\ &= \frac{\log 4}{\log 2} \cdot \frac{\log 8}{\log 4} \cdot \frac{\log 16}{\log 8} \cdot \frac{\log 32}{\log 16} \cdot \frac{\log 64}{\log 32} \cdot \frac{\log 128}{\log 64} \cdot \frac{\log 256}{\log 128} \cdot \frac{\log 512}{\log 256} \cdot \frac{\log 1024}{\log 512} \\ &= \frac{\log 1024}{\log 2} = \log_2 1024 = 10. \end{aligned}$$

47. There are five distinct positive integers whose mean is 10. What is the largest value of any one of these numbers?

Answer: 40 Note that since the mean of five integers is 10, the sum of those integers is 50. To get the largest possible integer, we should make the others as small as possible. Since the integers are distinct, the smallest distinct positive integers are 1, 2, 3, and 4. These sum to 10; this leaves 40 as the other number.

48. Suppose the real number k is the median of the set $\{9, k, -9, -1, 9, k^2, 1\}$. Then the interval $a \le k \le b$ represents all possibilities for k. Compute $b - a$.

Answer: 9 Since we do not know the value of k (or k^2), we must determine where they are to be inserted in the ordered list -9, -1, 1, 9, and 9. We consider all possible cases.

Case I: $k < 0$. The four numbers 1, 9, 9, and k^2 are positive, forcing the median to be 1, which is not k. So this case is not possible.

Case II: $k = 0$. Then the ordered list is -9, -1, 0, 0, 1, 9, 9, and the median is 0, which is k. So this case works.

Case III: $0 < k < 1$. Since $0 < k < 1$, then note that $k^2 < k$. Thus, we have an ordered list of -9, -1, k^2, k, 1, 9, 9, and the median is k. So this case works.

Case IV: $k = 1$. Then the ordered list is -9, -1, 1, 1, 1, 9, 9, and the median is 1, which is k. So this case works.

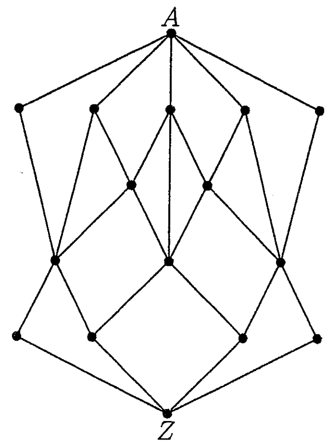
Case V: $1 < k < 9$. Note that $k < k^2$. Thus, we have an ordered list of -9, -1, 1, k, k^2, 9, 9, or possibly -9, -1, 1, k, 9, 9, k^2. In either list, the median is k. So this case works too.

Case VI: $k = 9$. Then the ordered list is -9, -1, 1, 9, 9, 9, 81, and the median is 9, which is k. So this case works.

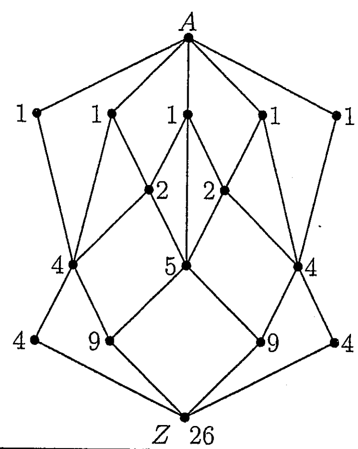
Case VII. $k > 9$. Then the ordered list is -9, -1, 1, 9, 9, k, k^2, and the median is 9, which is not k. So this case is not possible.

Finally, only Cases I and VII do not work, which restricts the possible values of k to those in the interval $0 \le k \le 9$. Thus, $b - a = 9 = 0 = 9$.

49. The graph below shows various routes (the edges) between a company's sixteen warehouses (the vertices). From the warehouse labeled A in the north to the warehouse labeled Z in the south, how many possible routes are there from A to Z traveling only in a southward direction?



Answer: 26 Working from top to bottom, we mark the path with the number of ways to get to each point by adding the possibilities from all of the points from which one travels. The marking is below, and there are 26 ways to get from A to Z by only traveling downward.



50. Ten thieves rob a wealthy entrepreneur. The thieves divvy out the spoils so that the first thief gets a tenth of the ill-gotten gains, which happens to be a hoard of identical gold coins. The second thief gets two-tenths of the remainder. The third thief gets three-tenths of that remainder, and so on (the nth thief gets n-tenths of currently remaining gold coins). Compute the smallest number of gold coins possible so that every thief gets a whole number of coins in their share.

Answer: 1562500 We focus on the tenth thief. Let G be the original amount of gold coins and let T be the tenth thief's share. The tenth thief's share is

$$T = \frac{1}{10}\left(\frac{2}{10}\left(\frac{3}{10}\left(\frac{4}{10}\left(\frac{5}{10}\left(\frac{6}{10}\left(\frac{7}{10}\left(\frac{8}{10}\left(\frac{9}{10}G\right)\right)\right)\right)\right)\right)\right)\right) = \frac{9!}{10^9}G.$$

Since G and T are integers, we may write

$$\begin{aligned} 9! \cdot G &= 10^9 \cdot T \\ 2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot G &= 2^9 \cdot 5^9 \cdot T \\ 3^4 \cdot 7 \cdot G &= 2^2 \cdot 5^8 \cdot T. \end{aligned}$$

Note that the primes on the left-hand side of the equation do not divide the primes on the right-hand side, and vice versa. This implies that the primes on the left must divide T, and the primes on the right must divide G. Hence, the smallest integer G must be equal to the primes on the right: $G = 2^2 \cdot 5^8 = (2 \cdot 5^4)^2 = 1250^2 = 1562500$. (And, the tenth thief got $T = 3^4 \cdot 7 = 567$ gold coins!)

CIPHERING SOLUTIONS

1. Solve: $x! = \dfrac{(5!)!}{5!}$.

Answer: 119 We have $\dfrac{(5!)!}{5!} = \dfrac{5!(5!-1)!}{5!} = (5!-1)! = (120-1)! = 119!$. Thus, $x = 119$.

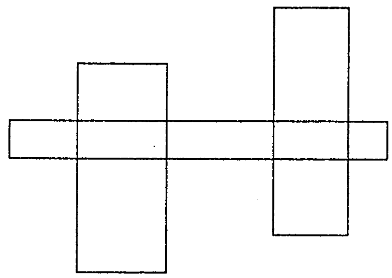
2. Suppose f and g are inverse functions, and $g(x) = x^3 - x$. Compute A so that $f(A) = 7$.

Answer: 336 Since f and g are inverses and $f(A) = 7$, then $g(7) = A$. Thus, $A = g(7) = 7^3 - 7 = 343 - 7 = 336$.

3. Let k and m be real numbers. The lines $y = kx + 1$ and $y = mx + 11$ intersect at the point $(5, 7)$. Compute the area of the region enclosed by the two lines and the y-axis.

Answer: 25 The y-intercepts of the two lines are $(0, 1)$ and $(0, 11)$, so consider this the base of the triangle; the base has length 10. The height of the triangle is now the distance from the y-axis to the intersection point $(5, 7)$; this is 5. Hence, the area is $\frac{1}{2} \cdot 10 \cdot 5 = 25$. (*Note:* The values of k and m are not necessary, but for completeness, they are $k = \frac{6}{5}$ and $m = -\frac{4}{5}$.)

4. Three rectangles overlap, as shown below. Compute the number of rectangles of all sizes which can be found in the figure.



Answer: 25 By brute force, we count. There are 9 non-overlapping rectangles in the figure. Now we consider combinations of these nine rectangles. There are 4 double rectangles, oriented horizontally; there are 4 double rectangles, oriented vertically; there are 3 triple rectangles, oriented horizontally; there are 2 triple rectangles, oriented vertically; there are 2 quadruple rectangles, both horizontally; and there is 1 rectangle made up of five smaller ones. The total is $9+4+4+3+2+2+1=25$.

5. Consider the function

$$f(x)=\frac{2x^2+x-7}{x-5}.$$

The oblique asymptote is $y=mx+b$ and the vertical asymptote is $x=c$. Compute $m+b-c$.

Answer: 8 The vertical asymptote is $x=5$. Using either polynomial long division or synthetic division to divide $2x^2+x-7$ by $x-5$, we find that the oblique asymptote is $y=2x+11$. Hence, $m+b-c=2+11-5=8$.

6. Convert the hexadecimal number 2016_{16} into base-10.

Answer: 8214 We have $2016_{16}=2\cdot 16^3+1\cdot 16+6=2\cdot 4096+16+6=8214$.

7. A regular hexadecagon (16-sided polygon) with a perimeter of 2016 is inscribed by a circle of radius r and is circumscribed by a circle of radius R. Compute the positive difference of the areas of the two circles.

Answer: 3969π The area of the large circle is πR^2 and the area of the small circle is πr^2. A hexadecagon has 16 sides, so a regular hexadecagon of perimeter 2016 has each side of length $2016/16=126$. Thus, we can split the hexadecagon into 16 congruent isosceles triangles with base 126, altitude r and congruent sides R. By the Pythagorean Theorem, $R^2-r^2=63^2=3969$, so the difference in areas is $\pi(R^2-r^2)=3969\pi$.

8. How many unique, nontrivial right triangles with integral side lengths have a leg of length 10?

Answer: 1 Call the other leg a and the hypotenuse c. Then $c^2 - a^2 = (c-a)(c+a) = 10^2 = 100$. Hence, $(c+a, c-a)$ is equal to $(10, 10)$, $(20, 5)$, $(25, 4)$, $(50, 2)$, or $(100, 1)$. Adding $c+a$ and $c-a$ together yields $2c = 20$, 25, 29, 52, or 101. The only values which make sense are the even ones: 20 and 52. However, if $2c = 20$, then $c = 10$, which implies $a = 0$, which is not viable. So it must be that $2c = 52$ so $c = 26$ and $a = 24$. Hence there is only one triangle which suits the criteria: the 10-24-26 right triangle.

9. Let k be an integer which satisfies the equation $\log_2(k-3) - \log_2(k+3) = -2$. Compute

$$\sum_{n=1}^{k} n^3.$$

Answer: 225 Combine logarithms to get $\log_2\left(\frac{k-3}{k+3}\right) = -2$, and then $\frac{k-3}{k+3} = 2^{-2} = \frac{1}{4}$. Solving this, we get $k = 5$. Finally,

$$\sum_{n=1}^{5} n^3 = 1 + 8 + 27 + 64 + 125 = 225.$$

10. Given $\sqrt{5 + 2\sqrt{6}} = \sqrt{x} + \sqrt{y}$, compute $|x - y|$.

Answer: 1 Note that

$$\sqrt{x} + \sqrt{y} = \sqrt{(\sqrt{x} + \sqrt{y})^2} = \sqrt{x + y + 2\sqrt{xy}}.$$

Thus, $\sqrt{x + y + 2\sqrt{xy}} = \sqrt{5 + 2\sqrt{6}}$, which requires $x + y = 5$ and $xy = 6$. Solving this system (probably fastest by inspection), we get one variable equal to 3, the other equal to 2. Therefore $|x - y| = 1$.

TEAM ROUND SOLUTIONS

1. Today is the 40th annual State Mathematics Tournament. Like today's event, the first State Mathematics Tournament also took place in Macon (at Northeast High School). More interestingly, like today's event, the first State Mathematics Tournament also took place on Saturday, April 30 (but, of course, in 1977, not 2016). Between 1977 and 2016 compute the sum of the years in which April 30 fell on a Saturday. (Do *not* include 1977 and 2016 in your sum.)

Answer: 9981 The simplest way to solve this problem is to make a table. As each year passes, April 30 will cycle through the days of the week, skipping over a day for each leap year. (Recall that if the year is divisible by 4 it is a leap year, unless it is divisible by 100 then it is not a leap year, but if it is divisible by 400, then it is a leap year.) We have the table below, starting with the fact that April 30 was a Saturday in 1977. The leap years are marked with an "L". The Saturdays are starred, indicating April 30 was a Saturday in the corresponding year.

1977	Sat	1990	Mon	2003	Wed
1978	Sun	1991	Tue	2004^L	Fri
1979	Mon	1992^L	Thu	2005	Sat*
1980^L	Wed	1993	Fri	2006	Sun
1981	Thu	1994	Sat*	2007	Mon
1982	Fri	1995	Sun	2008^L	Wed
1983	Sat*	1996^L	Tue	2009	Thu
1984^L	Mon	1997	Wed	2010	Fri
1985	Tue	1998	Thu	2011	Sat*
1986	Wed	1999	Fri	2012^L	Mon
1987	Thu	2000^L	Sun	2013	Tue
1988^L	Sat*	2001	Mon	2014	Wed
1989	Sun	2002	Tue	2015	Thu

We find the sum of the years with April 30 on a Saturday to be $1983+1988+1994+2005+2011 = 2000\cdot 5-17-12-6+5+11 = 10000-19 = 9981$.

2. Compute the area enclosed by the curves $y-2=-\frac{1}{3}(x+1)$ and $y=|x-1|$.

Answer: 2 Sketching the graphs reveals the region is a triangle. The sketch, or doing the algebra with the equations, also yields the intersection points $(-1,2)$ and $(2,1)$. The third point of the triangle is the vertex of the absolute value function, $(1,0)$. To compute the area, we use the "shoelace" method:

$$\text{Area} = \frac{1}{2}\begin{vmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 2 & 0\end{vmatrix} = \frac{1}{2}|1\cdot1+2\cdot2-1\cdot0-(0\cdot2-1\cdot1+2\cdot1)| = \frac{1}{2}|1+4-0-0+1-2| = \frac{1}{2}|4| = 2.$$

3. The line $y = x + m$ and the parabola $y = x^2 + n$ have one and only one point in common whenever m and n have the relationship $4m - 4n = k$. Compute k.

Answer: -1 FIRST (NON-CALCULUS) SOLUTION. Multiply both equations by 4 to get $4y = 4x + 4m$ and $4y = 4x^2 + 4n$. Then $k = 4m - 4n = 4y - 4x - (4y - 4x^2) = 4x^2 - 4x$. For there to be only one solution to $4x^2 - 4x - k = 0$, we must have (by completing the square) $k = -1$. (Indeed, this value gives $4x^2 - 4x + 1 = (2x - 1)^2 = 0$, which has only one solution.)

SECOND (CALCULUS) SOLUTION. With only one point in common, the line is tangent to the parabola, so the derivatives are equal at the point of intersection. Thus we require

$$\frac{d}{dx}(x + m) = \frac{d}{dx}(x^2 + n)$$

which implies $1 = 2x$, so $x = \frac{1}{2}$. But the y-values at that point are equal as well. Thus, $\frac{1}{2} + m = (\frac{1}{2})^2 + n$, which implies $2 + 4m = 1 + 4n$, or $4m - 4n = -1$.

4. Given the function $f(n) = (n+1)!(n-1)!$, where n is a positive integer, compute the smallest value of N such that $f(N)$ is divisible by 2187.

Answer: 10 Note that $2187 = 3^7$. So we are looking for at least seven powers of 3 in $f(N)$. Also, note that

$$f(N) = (N+1)!(N-1)! = (N+1)(N)(N-1)!(N-1)! = (N-1)!^2(N)(N+1),$$

and that $N-1$, N, and $N+1$ are three consecutive integers. Hence one of these three integers is divisible by 3. To get the most powers of three for the smallest N, we pick $N - 1 = 9$, since the term $(N-1)!$ is squared. Now, $9! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 3 \cdot 2 \cdot 7 \cdot 8 \cdot 3^2$ contains four powers of 3, so $(N-1)!^2$ contains eight powers of 3. With $N - 1 = 9$, $N = 10$ and $N + 1 = 11$, which contribute no powers of 3. Thus the we know that when $N = 10$, we get eight powers of 3. Is there a smaller value? We check $N = 9$. Then $(N-1)!^2(N)(N+1) = 8!^2 \cdot 9 \cdot 10 = (2 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 7 \cdot 8)^2(3^2)(10)$ has six powers of 3. So we indeed have that the smallest N which ensures divisibility by 2187 is $N = 10$.

5. Angela is in a room with three identical doors. She has equal probability of opening each of the doors. Behind the first door is a tiger. If she opens this door, there is a $\frac{4}{5}$ probability that the tiger will be hungry and eat her. Behind the second door is a giant snake. If she opens this door, there is a $\frac{2}{3}$ probability of being strangled by the snake. Behind the third door, there is a chocolate cake! If Angela opens this door, there is a 100% probability that she will eat the chocolate cake and survive. Given that Angela survives, the probability that she ate the cake is the reduced fraction $\frac{m}{n}$. Compute $m + n$.

Answer: 38 The probability of surviving is

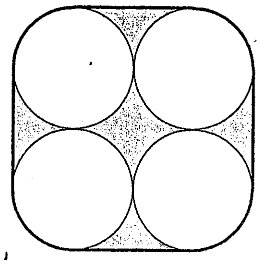
$$P(s) = \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot 1 = \frac{1}{15} + \frac{1}{9} + \frac{1}{3} = \frac{3 + 5 + 15}{45} = \frac{23}{45}.$$

The probability of surviving and eating cake is $P(c \cap s) = \frac{1}{3}$. Thus the probability of eating cake knowing that she survived is

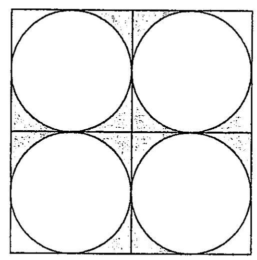
$$P(c \mid s) = \frac{P(c \cap s)}{P(s)} = \frac{\frac{1}{3}}{\frac{23}{45}} = \frac{15}{23}.$$

Finally, $m + n = 15 + 23 = 38$.

6. The diagram below shows four tangent unit circles with a rubber band stretched around them. Compute the area of the shaded region.

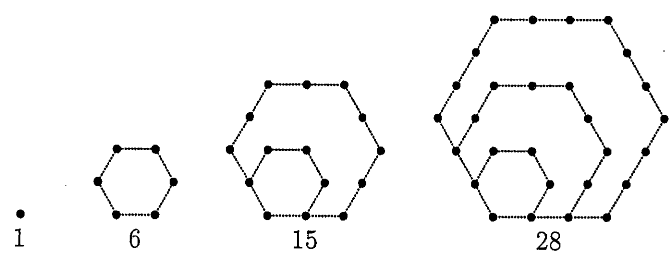


Answer: **$12-3\pi$** Draw a square around the figure and then subdivide the figure into four smaller squares, each externally tangent to a circle. Note that because the radius is 1, the side of a small square is 2.



Then we see that three-fourths of the area between a circle and a small square is shaded. Se we compute this area and multiply by 4. Hence, the shaded area is $4 \cdot \frac{3}{4}(2^2 - \pi) = 3(4 - \pi) = 12 - 3\pi$.

7. The first four hexagonal numbers are 1, 6, 15, and 28. Compute the 20th hexagonal number.

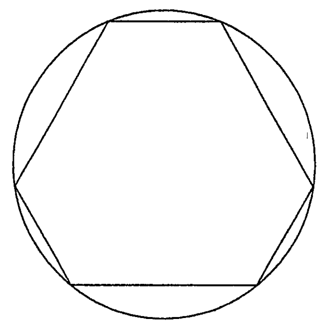


Answer: 780 We start by calculating the differences between the first four terms. We have the following.

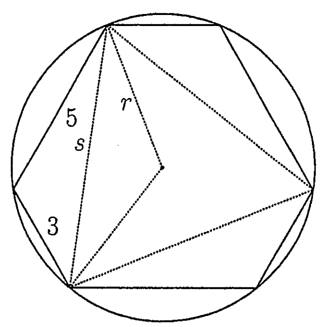
Terms	1		6		15		28
First Differences		5		9		13	
Second Differences			4		4		

Now, we could find the 20th hexagonal number by just continuing this pattern. However, it is more efficient to find a general formula. That the second difference is constant implies that the formula we seek is quadratic; thus, it can be written as $H_n = an^2 + bn + c$. The second difference of 4 tells us that $a = 2$. Now we can use the facts that $H_1 = 1$ and $H_2 = 6$ to find b and c. We have $H_1 = 2 \cdot 1^2 + b \cdot 1 + c = 1$, or $b + c = -1$; and $H_2 = 2 \cdot 2^2 + b \cdot 2 + c = 6$, or $2b + c = -2$. Therefore, $b = -1$ and $c = 0$ so that $H_n = 2n^2 - n$. Finally, $H_{20} = 2 \cdot 20^2 - 20 = 780$.

8. A hexagon with alternating side lengths of 3 and 5 is inscribed in a circle as shown. The radius of the circle can be written as the reduced fraction $\frac{m\sqrt{n}}{p}$, where n is not divisible by the square of any prime. Compute $m+n+p$.



Answer: 13 Because of the symmetry in the side lengths, this is an equiangular hexagon. Thus, like a *regular* hexagon, each angle is 120°. Then we can connect alternating vertices of the hexagon to form an equilateral triangle.



Now we use the Law of Cosines to calculate s^2.

$$s^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos 120° = 9 + 25 - 30\left(-\frac{1}{2}\right) = 34 + 15 = 49.$$

The radius of the circle can then be calculated again using the Law of Cosines. Again because of symmetry, the angle between the radii is also 120°. Hence,

$$s^2 = r^2 + r^2 - 2 \cdot r \cdot r \cdot \cos 120° = 2r^2 - 2r^2\left(-\frac{1}{2}\right) = 2r^2 + r^2 = 3r^2.$$

Therefore, $3r^2 = 49$, or $r = \sqrt{\frac{49}{3}} = \frac{7\sqrt{3}}{3}$, and $m+n+p = 7+3+3 = 13$.

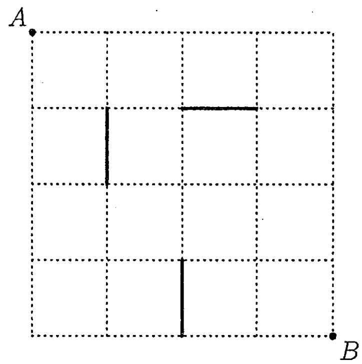
9. Completely simplify $\sqrt{15+6\sqrt{6}} + \sqrt{15-6\sqrt{6}}$.

Answer: 6 Let $x = \sqrt{15+6\sqrt{6}} + \sqrt{15-6\sqrt{6}}$, and then square both sides. We obtain

$$x^2 = 15 + 6\sqrt{6} + 2\sqrt{\left(15+6\sqrt{6}\right)\left(15-6\sqrt{6}\right)} + 15 - 6\sqrt{6},$$

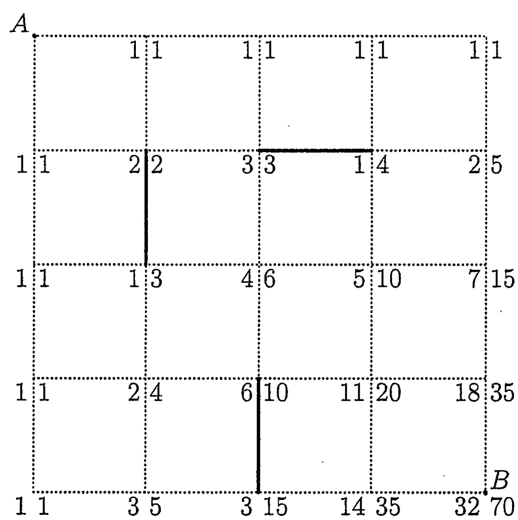
which simplifies to $x^2 = 30 + 2\sqrt{225-216} = 30 + 2\sqrt{9} = 36$. Therefore $x = 6$.

10. The following diagram represents a network of roads. Suppose one starts at point A, and randomly chooses to travel either down or to the right at each intersection until arriving at point B. However, there are three roads which must be avoided, shown in solid lines. The probability the three roads drawn with solid lines below are avoided can be written as the reduced fraction $\frac{p}{q}$, where p and q are integers. Compute $p + q$.



Answer: 51 The probability we seek is the number of ways we can get from A to B avoiding the solid-marked roads divided by the number of ways one can simply get from A to B. To find the latter, note there is only one way to get to any of the intersections along the top or left-most roads. Given that the only way to get to any other intersection is from above or from the left, to find the number of ways we can get to intersection X we sum the number of ways to get to the intersections above and to the left of X. These sums can be found incrementally starting at A and working our way down and to the right and are shown to the *right* of each intersection below.

Finding the number of ways we can get from A to B avoiding the solid-marked roads is calculated similarly, except in a small number of cases. As an example, the number of ways we can get to the middle of the bottom row is identical to the number of ways we can get to the intersection to its immediate left. The intersection above this middle intersection doesn't contribute to the aforementioned sum, as there is a road between them to be avoided. These ultimately smaller sums are also quickly calculated and shown to the *left* of each intersection below.



So the probability sought is $\frac{32}{70} = \frac{16}{35}$. Thus $p + q = 16 + 35 = 51$.

11. A digit x is called *twitchy* in base b if $x \neq 1$ and x divides an integer m whenever x divides the sum of the digits of m written in base b. As an example, both 3 and 9 are twitchy in base 10. What is the sum, in base 10, of all twitchy digits in base 7?

Answer: 11 Since 1 is never twitchy and 0 doesn't affect the requested sum, there are only really five digits to consider: 2, 3, 4, 5, and 6. The base 10 number 10 when written in base 7 is 13; 4 divides the sum of the digits, but does not divide 10. Thus, 4 is not twitchy. Similarly, the base 10 number 11 when written in base 7 is 14. 5 divides the sum of the digits, but does not divide 11. Thus, 5 is not twitchy.

Considering a base 10 number like 240 that when written in base 7 is 462, we see that 2, 3, and 6 all divide the sum of the digits, and the original number. So our limited experimentation suggests that 2, 3, and 6 *might be* twitchy in base 7. Taking the risk of going with one's intuition on this question does indeed give the correct answer of $2 + 3 + 6 = 11$.

That said, let's argue more rigorously.

Recall, if x divides $\sum_{i=0}^{n} d_i 7^n$ whenever x divides $\sum_{i=0}^{n} d_i$, then x is twitchy in base 7. As such, if we could show $\sum_{i=0}^{n} d_i 7^n - \sum_{i=0}^{n} d_i$ were always divisible by x (or equivalently, if $\sum_{i=0}^{n} d_i(7^n - 1)$ were always divisible by x) then x would necessarily be twitchy in base 7. (*Note:* When x divides one of the two terms and the entire difference, it must divide the other term.)

Along these lines—and noting that the digits d_i could be anything—we focus on the $(7^n - 1)$ above. if we are lucky enough that we can show x divides $7^n - 1$ for all integers n, then x must be twitchy in base 7.

Let us consider $x = 6$ in detail. To prove 6 divides $7^n - 1$ for any positive integer n, we proceed inductively. $7^1 - 1 = 6$ which 6 obviously divides. This establishes the base case. Assume 6 divides $7^k - 1$ for an integer k. Note that $7^{k+1} - 1 = 6 \cdot 7^k + (7^k - 1)$. 6 divides both terms, so it must divide their sum. This establishes the inductive case.

We can argue in a parallel way to prove 2, 3, and 6 all divide $7^n - 1$ for any positive integer n. As such, 2, 3, and 6 are all twitchy in base 7. So the sum of all twitchy digits in base 7 is indeed $2 + 3 + 6 = 11$.

12. A game of Craps is played in the following way: A pair of dice is rolled. If the roll results in a 7 or 11, the player wins. If the roll results in a 2, 3, or 12, the player loses. If any other roll results, the value of that roll is recorded as the *point* and then the dice are rolled repeatedly until either a 7 or the *point* value is rolled. If the *point* value is rolled before a 7 is rolled, the player wins. If a 7 is rolled before the *point* value was rolled, the player loses. If the probability of winning this game is the reduced fraction $\frac{p}{q}$, compute $p+q$.

Answer: 739 Consider the sample space for the dice rolls below.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

We will require two types of probabilities: the probability of rolling a sum of x in a single roll; and the probability of rolling an x before a y. The first case is immediate. Consider $x = 4$. There are three 4's in the sample space of 36 equally likely events, so the probability is $\frac{3}{36}$. In the second case, note that re-rolling upon not getting an x or y effectively reduces the sample space to only x's and y's. As an example – given there are nine 4's and 7's total, but only three 4's, the probability of rolling a 4 before a 7 is $\frac{3}{9}$.

Armed with these, we can now find the probability of winning the game. Let $P(X)$ be the probability of event X happening. We have

$$\begin{aligned} P(\text{win on 1st roll}) &= P(7 \text{ on first roll}) + P(11 \text{ on first roll}) \\ &= \frac{6}{26} + \frac{2}{36} = \frac{8}{36}. \end{aligned}$$

Next,

$$\begin{aligned} P(\text{win on 2nd roll}) &= P(4 \text{ on first roll}) \times P(4 \text{ rolled before } 7) \\ &\quad + P(5 \text{ on first roll}) \times P(5 \text{ rolled before } 7) \\ &\quad + P(6 \text{ on first roll}) \times P(6 \text{ rolled before } 7) \\ &\quad + P(8 \text{ on first roll}) \times P(8 \text{ rolled before } 7) \\ &\quad + P(9 \text{ on first roll}) \times P(9 \text{ rolled before } 7) \\ &\quad + P(10 \text{ on first roll}) \times P(10 \text{ rolled before } 7) \\ &= \frac{3}{36}\cdot\frac{3}{9} + \frac{4}{36}\cdot\frac{4}{10} + \frac{5}{36}\cdot\frac{5}{11} + \frac{5}{36}\cdot\frac{5}{11} + \frac{4}{36}\cdot\frac{4}{10} + \frac{3}{36}\cdot\frac{3}{9} \\ &= \frac{\left(\frac{9}{9} + \frac{16}{10} + \frac{25}{11} + \frac{25}{11} + \frac{16}{10} + \frac{9}{9}\right)}{36} \\ &= \frac{134}{495}. \end{aligned}$$

So, $P(\text{win}) = P(\text{win on 1st roll}) + P(\text{win on 2nd roll}) = \frac{8}{36} + \frac{134}{495} = \frac{244}{495}$, and thus, $p+q = 244+495 = 739$.