2013

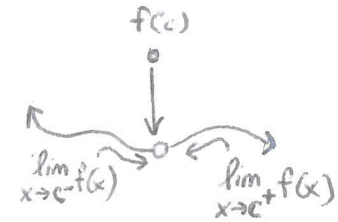
AP Calculus AB Trig Unit Test Review WS #1 Name:______________________

Continuity review: 3 conditions for continuity at a point:

1) f(c) is defined

2) $\lim\limits_{x\to c} f(x)$ exists $\left(\text{This means that } \lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)\right)$

3) $\lim\limits_{x\to c} f(x) = f(c)$



1. Find the constant **a** such that the function is continuous at the given point.

a) $g(x) = \begin{cases} \dfrac{7\sin(2x)}{4x}, & x > 0 \\ a - 2x, & x \le 0 \end{cases}$

$\lim\limits_{x\to 0^+} \dfrac{7\sin(2x)}{4x} = \dfrac{7(2)}{4} = \dfrac{7}{2}$

$\lim\limits_{x\to 0^-} a - 2x = a$

$\boxed{a = 7/2}$

$g(x) = \begin{cases} \dfrac{7\sin 2x}{4}, & x > 0 \\ 7/2 - 2x, & x \le 0 \end{cases}$

i) $g(0) = 7/2$

ii) $\lim\limits_{x\to 0} g(x) = 7/2$

iii) $\lim\limits_{x\to 0} g(x) = g(0) = 7/2$ ✓

b) $g(x) = \begin{cases} \dfrac{1-\cos(2x)}{2x} - a, & x > 0 \\ \cos x + a, & x \le 0 \end{cases}$

$\lim\limits_{x\to 0^+} \dfrac{1-\cos(2x)}{2x} - a = 0 - a = -a$

$\lim\limits_{x\to 0^-} \cos x + a = \cos 0 + a = 1 + a$

$1 + a = -a$

$1 = -2a$

$\boxed{-\tfrac{1}{2} = a}$

c) $g(x) = \begin{cases} \tan\dfrac{x}{4}, & x > \pi \\ -2\sec x + a, & x \le \pi \end{cases}$

$\lim\limits_{x\to \pi^+} \tan\frac{x}{4} = \tan(\pi/4) = 1$

$\lim\limits_{x\to \pi^-} -2\sec x + a = -2\sec\pi + a = -2(-1) + a = 2 + a$

$2 + a = 1$

$\boxed{a = -1}$

2. Determine whether Rolle's Theorem can be applied to $f(x) = \sec x$ on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. If so, find all values of c in the interval such that $f'(c) = 0$. ① $f(x)$ continuous on $[-\pi/4, \pi/4]$ and ② differentiable on $(-\pi/4, \pi/4)$

$f(a) = f(b)$?

③ $f(-\pi/4) = \sec(-\pi/4) = \sqrt{2}$

$f(\pi/4) = \sec(\pi/4) = \sqrt{2}$

set $f'(x) = 0$

$f'(x) = \sec x \tan x$

$\sec x \tan x = 0$

$\dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = 0$

$\dfrac{\sin x}{\cos^2 x} = 0$

$\sin x = 0$

$x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$

check interval

$x = 0$

$\boxed{c = 0}$

3. Apply the Mean Value Theorem to $f(x) = 2\sin x + \sin 2x$ on the interval $[0, \pi]$. Find all values of c in the interval $(0, \pi)$. ① $f(x)$ continuous on $[0, \pi]$ and ② differentiable on $(0, \pi)$

set $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

$f(0) = 2\sin(0) + \sin(2(0)) = 0$

$f(\pi) = 2\sin(\pi) + \sin 2\pi = 0$

$\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{0 - 0}{\pi} = 0$

$f'(x) = 2\cos x + \cos(2x) \cdot 2$

$2\cos x + 2\cos(2x) = 0$

$2\cos x + 2[2\cos^2 x - 1] = 0$

$4\cos^2 x + 2\cos x - 2 = 0$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = 1/2$ | $\cos x = -1$

$x = \pi/3, \cancel{5\pi/3}$ | $x = \cancel{\pi}$

$x = \pi/3$

$\boxed{c = \pi/3}$

4. An object's position is given by the $F(t) = 2\sec^3\left(\frac{t}{6}\right)$, is continuous and differentiable in domain $0 \le t < 3\pi$ seconds. F(t) is given in meters.

a. Find the average velocity (avg. rate of change) from $t = 0$ to $t = 2\pi$ (Include Units)

*This is simply finding slope between endpoints. No derivatives involved.

$f(0) = 2\sec^3(0) = 2(1) = 2$

$f(2\pi) = 2\sec^3(2\pi/6) = 2[\sec(\pi/3)]^3$

$= 2[2]^3$

$= 16$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2\pi)-f(0)}{2\pi-0} = \frac{16-2}{2\pi-0} = \frac{14}{2\pi} = \boxed{\frac{7}{\pi}\ m/s}$$

b. At what point in time does the instantaneous velocity equal the average velocity from part (a)? (Set up equation but do not solve) Use MVT.

set $f'(x) = \frac{f(b)-f(a)}{b-a}$

$f(x) = 2[\sec(t/6)]^3$

$f'(x) = 2\cdot 3[\sec(t/6)]^2 \cdot \sec(t/6)\tan(t/6)\cdot\frac{1}{6}$

$= \sec^3(t/6)\tan(t/6)$

$$\boxed{\sec^3(t/6)\tan(t/6) = \frac{7}{\pi}}$$

c. What is the instantaneous velocity of the object when $t = \pi$ seconds? (Include Units)

$f'(x) = \sec^3(t/6)\tan(t/6)$

$= \frac{1}{\cos^3(t/6)}\cdot\frac{\sin(t/6)}{\cos(t/6)}$

$f'(x) = \frac{\sin(t/6)}{\cos^4(t/6)}$

$$f'(\pi) = \frac{\sin(\pi/6)}{\cos^4(\pi/6)} = \frac{(1/2)}{(\sqrt{3}/2)^4} = \frac{\frac{1}{2}}{\frac{9}{16}} = \frac{1}{2}\cdot\frac{16}{9} = \boxed{8/9\ m/s}$$

d. What is the instantaneous velocity of the object when $t = 2\pi$ seconds? (Include Units)

$$f'(2\pi) = \frac{\sin(2\pi/6)}{\cos^4(2\pi/6)} = \frac{\sin(\pi/3)}{\cos^4(\pi/3)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{(1/2)^4} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{16}} = \frac{\sqrt{3}}{2}\cdot\frac{16}{1} = \boxed{8\sqrt{3}\ m/s}$$

e. Find the equation of the tangent line to the graph at $t = 2\pi$

$$f(2\pi) = 2\sec^3\left(\frac{2\pi}{6}\right) = 2\sec^3(\pi/3) = \frac{2}{\cos^3(\pi/3)} = \frac{2}{(1/2)^3} = \frac{2}{1/8} = 2\cdot 8 = 16$$

$f'(2\pi) = 8\sqrt{3}$

point: $(2\pi, 16)$

slope: $m = 8\sqrt{3}$

$y - y_1 = m(x - x_1)$

$$\boxed{y - 16 = 8\sqrt{3}(x - 2\pi)}$$

AP Calculus AB Trig Unit Test Review WS #2 Name________________

1. Solve for x: cos(2x) = 1 for $0 < x < 2\pi$

$(2x) = \cos^{-1}(1)$

$2x = 0, 2\pi, 4\pi, 6\pi, 8\pi, \ldots$

$x = \frac{0}{2}, \frac{2\pi}{2}, \frac{4\pi}{2}, \frac{6\pi}{2}, \frac{8\pi}{2}$

$x = \cancel{0}, \pi, \cancel{2\pi}, \cancel{3\pi}$

$\boxed{x = \pi}$

2. Solve for x: tan(4x) = 1 for $0 < x < \pi$

$(4x) = \tan^{-1}(1)$

$= \pi/4, 5\pi/4, 9\pi/4, 13\pi/4, 17\pi/4$

$x = \frac{\pi}{4}\cdot\frac{1}{4}, \frac{5\pi}{4}\cdot\frac{1}{4}, \frac{9\pi}{4}\cdot\frac{1}{4}, \frac{13\pi}{4}\cdot\frac{1}{4}, \frac{17\pi}{4}\cdot\frac{1}{4}$

$x = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \cancel{\frac{17\pi}{16}}$

$\boxed{x = \pi/16, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}}$

3. Find the critical points for the function $e^{x + \sin 2x}$ in the domain $0 < x < 2\pi$

$y' = e^{x+\sin 2x} \cdot (1 + 2\cos(2x))$

$e^{x+\sin 2x} \neq 0$

$1 + 2\cos(2x) = 0$

$2\cos(2x) = -1$

$\cos(2x) = -1/2$

$2x = \cos^{-1}(-1/2)$

$2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$

$\boxed{x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3}$

OR

$1 + 2\cos(2x) = 0$

$1 + 2(2\cos^2 x - 1) = 0$

$1 + 4\cos^2 x - 2 = 0$

$4\cos^2 x = 1$

$\cos^2 x = 1/4$

$\cos x = \pm 1/2$

$\boxed{x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3}$

4. Find the equation of the line tangent to the graph at (3,-1) for the equation $\sin(\pi x) + \cos(\pi y) = x^2 y$

* Apply implicit differentiation, chain, trig, product rules.

plug in (3,-1) to find $\frac{dy}{dx}$

$\cos(\pi x)\cdot\pi - \sin(\pi y)\cdot\pi\left(\frac{dy}{dx}\right) = 2xy + x^2\left(\frac{dy}{dx}\right)$

$\pi\cos(\pi x) - \pi\sin(\pi y)\frac{dy}{dx} = 2xy + x^2\left(\frac{dy}{dx}\right)$

$\pi\cos(3\pi) - \pi\sin(-\pi)\frac{dy}{dx} = 2(3)(-1) + 3^2\frac{dy}{dx}$

$\pi(-1) - \cancel{\pi(0)\frac{dy}{dx}} = -6 + 9\frac{dy}{dx}$

$-\pi = -6 + 9\frac{dy}{dx}$

$\frac{-\pi + 6}{9} = \frac{dy}{dx}$

point: $(3,-1)$

slope: $m = \frac{-\pi + 6}{9}$

$y - y_1 = m(x - x_1)$

$\boxed{y + 1 = \frac{6-\pi}{9}(x - 3)}$

5. Person X and Person Y are walking on straight streets that meet at right angles. Y travels south and approaches the intersection at 2m/s. Person X travels east and moves away from the intersection at 1m/s.
   a) Find the rate at which the distance (Z) between Person X and Y is changing when Y is 10m from the intersection and X is 20 meters from the intersection.
   b) At what rate is the angle θ changing at the same moment?

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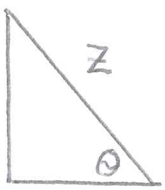


$x = 20m$ $\quad dx/dt = 1\,m/s$

$y = 10\,m$ $\quad dy/dt = -2m/s$

$z = 10\sqrt{5}\,m$ $\quad dz/dt = \_\_\_\_$

$10^2 + 20^2 = z^2$

$z = 10\sqrt{5}$

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

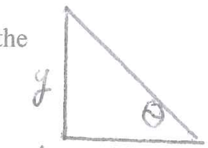
$$2(20)(1) + 2(20)(-2) = 2(10\sqrt{5})\frac{dz}{dt}$$

$$40 - 40 = 20\sqrt{5}\frac{dz}{dt}$$

$$\frac{0}{20\sqrt{5}} = \frac{20\sqrt{5}}{20\sqrt{5}}\frac{dz}{dt}$$

$$\boxed{\frac{dz}{dt} = 0\,m/s}$$

$$\tan\theta = \frac{y}{x}$$



$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{(dy/dt)(x) - y(dx/dt)}{x^2}$$

$$\frac{d\theta}{dt} = \frac{(dy/dt)(x) - y(\frac{dx}{dt})}{x^2 \cdot \sec^2\theta} = \frac{(-2)(20) - (10)(1)}{20^2 \cdot \left(\frac{10\sqrt{5}}{20}\right)^2}$$

$$= \frac{-50}{20^2 \cdot \frac{10\sqrt{5}^2}{20^2}} = \frac{-50}{(10\sqrt{5})^2} = \frac{-50}{500}$$

$$\boxed{\frac{d\theta}{dt} = \frac{-1}{10}\,rad/s}$$

6. Sketch the graph if function $f(x) = x - \sin x$ on the interval $[0, 4\pi]$. Find all ordered pairs of absolution and relative extrema and POI

$f'(x) = 1 - \cos x$ $\quad x = 0, 2\pi, 4\pi$

$0 = 1 - \cos x$

$\cos x = 1$

$0 \quad + \quad 2\pi \quad + \quad 4\pi$ (↗ ↗)

$f''(x) = 0 + \sin x$

$\sin x = 0$

$x = 0, \pi, 2\pi, 3\pi, 4\pi$

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$0 \quad + \quad \pi \quad - \quad 2\pi \quad + \quad 3\pi \quad - \quad 4\pi$

a) the relative extrema min: none max: none

$f(\pi) = \pi - \sin\pi = \pi$

$f(2\pi) = 2\pi - \sin(2\pi) = 2\pi$

$f(3\pi) = 3\pi - \sin(3\pi) = 3\pi$

$f(4\pi) = 4\pi - \sin(4\pi) = 4\pi$

b) the point(s) of inflection: $(\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi)$ b/c $f''(x)$ change signs

c) concavity intervals

up: $(0, \pi) \cup (2\pi, 3\pi)$ b/c $f''(x) > 0$

down: $(\pi, 2\pi) \cup (3\pi, 4\pi)$ b/c $f''(x) < 0$

d) Absolute extrema: Absolute min: $(0, 0)$ Absolute max: $(4\pi, 4\pi)$

e) sketch the graph

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$0 \quad \pi \quad 2\pi \quad 3\pi \quad 4\pi$