Trig Limits (1.3, 1.4, 3.5)

(1.3) p. 67-68 #27-35 odd
#63-73 odd
(1.4) p. 80 #21, 55, 57, 76
(3.5) p. 202-203 #5, 35, 37, 43

Ch. 1.3

*Recall: $\lim\limits_{x\to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}$ $\lim\limits_{x\to 0} \frac{1-\cos(ax)}{(ax)} = 0$

63) $\lim\limits_{x\to 0} \frac{\sin x}{5x} = \boxed{\frac{1}{5}}$

65) $\lim\limits_{x\to 0} \frac{\sin x(1-\cos x)}{x^2} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \frac{(1-\cos x)}{x} = 1 \cdot 0 = \boxed{0}$

67) $\lim\limits_{x\to 0} \frac{\sin^2 x}{x} = \lim\limits_{x\to 0} \overbrace{\boxed{\frac{\sin x}{x}}}^{=1} \cdot \sin x = \lim\limits_{x\to 0} \sin x = \sin 0 = \boxed{0}$

69) $\lim\limits_{h\to 0} \frac{(1-\cos h)^2}{h} = \lim\limits_{h\to 0} \overbrace{\boxed{\frac{(1-\cos h)}{h}}}^{=0} \cdot (1-\cos h) = \lim\limits_{h\to 0} 0 \cdot (1-\cos h) = 0 \cdot 0 = \boxed{0}$

71) $\lim\limits_{x\to \pi/2} \frac{\cos x}{\cot x} = \lim\limits_{x\to \pi/2} \frac{\cos x \cdot \frac{\sin x}{\cos x}}{\frac{\cos x}{\sin x} \cdot \frac{\sin x}{\cos x}} = \lim\limits_{x\to \pi/2} \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} = \lim\limits_{x\to \pi/2} \sin x = \sin\left(\frac{\pi}{2}\right) = \boxed{1}$

73) $\lim\limits_{t\to 0} \frac{\sin 3t}{2t} = \boxed{\frac{3}{2}}$

Ch. 1.4 p. 80 # 21, 55, 57, 76
#56

Recall continuity conditions:
i) $f(c)$ exists
ii) $\lim_{x\to c} f(x)$ exists $\left[\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)\right]$
iii) $f(c) = \lim_{x\to c} f(x)$

55) $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases}$ Find x-values where f is not continuous

$f(x) = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1,\ x \geq 1 \end{cases}$ * Test possible discontinuities at $x = 1$, $x = -1$

* step through continuity conditions:

Test $x = -1$

a) $f(-1) = -1$

b) $\lim_{x\to -1^-} x = -1$ $\lim_{x\to -1^+} \tan\left(\frac{\pi x}{4}\right) = \tan\left(\frac{-\pi}{4}\right) = -1$

c) $f(-1) = \lim_{x\to -1} f(x) = -1$

f(x) continuous at $x = -1$

Test $x = 1$

a) $f(1) = 1$

b) $\lim_{x\to 1^-} \tan \frac{\pi x}{4} = 1$, $\lim_{x\to 1^+} x = 1$

c) $f(1) = \lim_{x\to 1} f(x) = 1$ ✓

f(x) continuous at $x = 1$

* f(x) continuous for all real x.

56) $f(x) = \begin{cases} \csc \frac{\pi x}{6}, & |x-3| \leq 2 \\ 2, & |x-3| > 2 \end{cases}$ $\to -2 < x-3 < 2 \to 1 \leq x \leq 5$; $\to x < 1,\ x > 5$

* Test possible discontinuities at $x = 1$, $x = 5$

Test $x = 1$

a) $f(1) = \csc \frac{\pi}{6} = 2$

b) $\lim_{x\to 1^+} \csc \frac{\pi x}{6} = 2$, $\lim_{x\to 1^-} 2 = 2$

c) $f(1) = \lim_{x\to 1} f(x) = 2$ ✓

Test $x = 5$

a) $f(5) = \csc\left(\frac{5\pi}{6}\right) = 2$

b) $\lim_{x\to 5^-} \csc \frac{5\pi}{6} = 2$, $\lim_{x\to 5^+} 2 = 2$

c) $f(5) = \lim_{x\to 5} f(x) = 2$ ✓

$\boxed{\text{f(x) is continuous for all real x}}$

57) $f(x) = \csc 2x$ Nonremovable discontinuities at integer multiples of $\pi/2$

$$f(x) = \frac{1}{\sin(2x)} \to \frac{1}{\sin 2\cdot \pi/2} \to \frac{1}{\sin \pi}$$

76) $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$ * Test possible discontinuity at $x=0$

i) $f(0) = 5(0) = 0$

ii) $\lim_{x\to 0^-} \frac{\cos x - 1}{x} = \lim_{x\to 0^-} \frac{-(1-\cos x)}{x} = 0$ $\lim_{x\to 0^+} 5x = 0$

iii) $f(0) = \lim_{x\to 0} f(x) = 0$

$f(x)$ is continuous for all real x.

$\boxed{3.5}$ p.202-203 #5, 35, 37, 43

35) $\lim_{x\to\infty} \frac{1}{2x+\sin x} = 0$

37) $\lim_{x\to\infty} \frac{\sin 2x}{x}$ since $\frac{-1}{x} \leq \frac{\sin 2x}{x} \leq \frac{1}{x}$ * Squeeze Theorem

$\lim_{x\to\infty} \frac{-1}{x} = 0$ $\lim_{x\to\infty} \frac{1}{x} = 0$, therefore by squeeze theorem

$\lim_{x\to\infty} \frac{\sin 2x}{x} = 0$.

43) $\lim_{x\to\infty} x \sin \frac{1}{x}$ * let $x = \frac{1}{t}$

$$\lim_{t\to 0^+} \frac{1}{t}\cdot \sin t = \lim_{t\to 0^+} \frac{\sin t}{t} = \boxed{1}$$

Trig: Ch. 2.3 p.125-126 #11, 39, 41, 43, 51, 53

Ch. 2.4 p.136-139 #47, 49, 53-63 odd, 71, 77a, #79, 89

2.4 Find derivative *Recall chain rule: $\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x)$

47) $y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$

$y' = \cos(\pi^2 x^2) \cdot \pi^2 2x = \boxed{2\pi^2 x \cos(\pi^2 x^2)}$

49) $h(x) = \sin 2x \cos 2x$ *Apply product rule

$h'(x) = \underbrace{\cos 2x (2)}_{f'} \cdot \underbrace{\cos 2x}_{g} + \underbrace{\sin(2x)}_{f} \cdot \underbrace{(-\sin(2x)) \cdot 2}_{g'}$

$= 2\cos^2(2x) - 2\sin^2(2x)$ *Recall $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2(2\theta) = \cos^2(2\theta) - \sin^2(2\theta)$

$= 2\cos 2(2x)$

$= \boxed{2\cos 4x}$

53) $y = 4\sec^2 x = 4(\sec x)^2$

$y' = 4 \cdot 2(\sec x)^1 \cdot \sec x \tan x = \boxed{8\sec^2 x \tan x}$

55) $f(\theta) = \tan^2(5\theta) = [\tan(5\theta)]^2$ *chain rule $\frac{d}{dx} f[g(h(x))] = f'[g(h(x))] \cdot g'(h(x)) \cdot h'(x)$

$f'(\theta) = 2[\tan(5\theta)]^1 \sec^2(5\theta) \cdot 5$

$= \boxed{10\tan(5\theta)\sec^2(5\theta)}$

57) $f(\theta) = \frac{1}{4}\sin^2(2\theta) = \frac{1}{4}[\sin(2\theta)]^2$

$f'(\theta) = \frac{1}{4} \cdot 2[\sin(2\theta)]^1 \cos(2\theta) \cdot 2$

$\boxed{f'(\theta) = \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 2(2\theta) = \frac{1}{2}\sin 4\theta}$

$\boxed{2.4}$

59) $f(t) = 3\sec^2(\pi t - 1) = 3\left[\sec(\pi t - 1)\right]^2$

$f'(t) = 3 \cdot 2\left[\sec(\pi t-1)\right]' \sec(\pi t - 1)\tan(\pi t - 1) \cdot \pi$

$\boxed{f'(t) = 6\pi \sec^2(\pi t - 1)\tan(\pi t - 1)}$

61) $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2 = x^{1/2} + \frac{1}{4}\sin(4x^2)$

$y' = \frac{1}{2}x^{-1/2} + \frac{1}{4}\cos(4x^2) \cdot 8x = \boxed{\frac{1}{2\sqrt{x}} + 2x\cos(4x^2)}$

63) $y = \sin(\tan 2x)$

$y' = \cos(\tan 2x) \cdot \sec^2(2x) \cdot 2 = \boxed{2\cos(\tan 2x) \cdot \sec^2(2x)}$

71) Evaluate derivative at a point $y = 26 - \sec^3 4x$ at $(0, 25)$

$y = 26 - \left[\sec 4x\right]^3$

$y' = 0 - 3\left[\sec 4x\right]^2 \sec(4x)\tan(4x) \cdot 4$

$y' = -12\sec^3 4x \tan 4x$

$y'(0) = -12\left[\sec 0\right]^3\left[\tan 0\right] = \boxed{0}$

77) Find equation of tangent line

$f(x) = \sin 2x$ at $(\pi, 0)$

$f'(x) = \cos 2x \cdot 2$

$f'(\pi) = 2\cos(2\pi) = 2(1) = 2$

point: $(\pi, 0)$ slope: $m = 2$

$\boxed{y - 0 = 2(x - \pi)}$

79) Find equation of tangent line

$f(x) = \tan^2 x$ at $(\pi/4, 1)$

$f(x) = \left[\tan x\right]^2$

$f'(x) = 2\left[\tan x\right]' \sec^2 x$

$f'(\pi/4) = 2\tan(\pi/4)\left[\sec \pi/4\right]^2 = 2(1)\left(\frac{2}{\sqrt{2}}\right)^2$

$f'(\pi/4) = 2 \cdot \left(\frac{4}{2}\right) = 4$

point: $(\pi/4, 1)$ slope: $m = 4$

$\boxed{y - 1 = 4(x - \pi/4)}$

89) Find $f''(x)$ $\quad f(x) = \sin x^2$

$$f'(x) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

$$f''(x) = 2\cos(x^2) + 2x \cdot (-\sin x^2) \cdot 2x$$

$$\boxed{f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)}$$

Ch. 2.5 (Trig) p.145-146 #11-15 odd, 27

Implicit differentiation: Find $\frac{dy}{dx}$

11) $\sin x + 2\cos 2y = 1$

$$\cos x + 2(-\sin 2y) \cdot 2\left(\frac{dy}{dx}\right) = 0$$

$$-4\sin 2y\left(\frac{dy}{dx}\right) = -\cos x$$

$$\boxed{\frac{dy}{dx} = \frac{\cos x}{4\sin 2y}}$$

*product rule

13) $\sin x = x(1 + \tan y)$

$$\cos x = 1(1 + \tan y) + x\left[\sec^2 y\right]\left(\frac{dy}{dx}\right)$$

$$\cos x - 1 - \tan y = x\sec^2 y\left(\frac{dy}{dx}\right)$$

$$\boxed{\frac{\cos x - 1 - \tan y}{x\sec^2 y} = \frac{dy}{dx}}$$

15) $y = \sin xy$

*chain rule
*product rule

$$\frac{dy}{dx} = \cos(xy) \cdot \left[1y + x\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = y\cos xy + x\cos(xy)\frac{dy}{dx}$$

$$\frac{dy}{dx} - \frac{dy}{dx}x\cos xy = y\cos xy$$

$$\frac{dy}{dx}(1 - x\cos xy) = y\cos xy$$

$$\boxed{\frac{dy}{dx} = \frac{y\cos xy}{1 - x\cos xy}}$$

27) Evaluate derivative at a given point

$\tan(x+y) = x$ at $(0,0)$

$\sec^2(x+y)\left[1+\frac{dy}{dx}\right] = 1$

$\sec^2(0+0)\left[1+\frac{dy}{dx}\right] = 1$

$(1^2)\left[1+\frac{dy}{dx}\right] = 1$

$1+\frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = 0}$