Key

10.3 AP Practice Problems (p. 746) - Tests for Convergence (nth term, p-series, Integral Test)

1. For what numbers p does the series $\sum_{k=1}^{\infty} \frac{1}{k^{p/3}}$ converge?

(A) $p > 1$ (B) $|p| > 1$ $\boxed{\text{(C)}}$ $p > 3$ (D) $p > \frac{1}{3}$

p-series

$\frac{p}{3} > 1$

$p > 3$

2. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \frac{e^{k-1}}{3^{k-1}}$ II. $\sum_{k=1}^{\infty} \cos\left(\pi + \frac{1}{k}\right)$ III. $\sum_{k=1}^{\infty} \frac{10}{k}$ → harmonic series

(A) I and II only (B) I and III only

$\boxed{\text{(C)}}$ II and III only (D) I, II, and III

i) $\frac{e^k \cdot e^{-1}}{3^k \cdot 3^{-1}} \to \frac{3}{e}\left(\frac{e}{3}\right)^k$

$\boxed{\text{converges}}$ $r = \frac{e}{3} < 1$ ✓

ii) $\cos\pi \cos\left(\frac{1}{k}\right) - \sin\pi \sin\left(\frac{1}{k}\right)$

$-\cos\left(\frac{1}{k}\right) - 0$

diverges

3. For what values of p does the series $\sum_{k=1}^{\infty} k^p$ converge?

(A) $-1 < p < 1$ $\boxed{\text{(B)}}$ $p < -1$

(C) $p > 1$ (D) The series diverges.

$\sum \frac{1}{k^{-p}}$ if $-p > 1$

$p < -1$

4. $\sum_{k=1}^{\infty} \frac{5^{k-1} - 3^{k-1}}{8^{k-1}} =$

(A) $\frac{1}{3}$ $\boxed{\text{(B)}}$ $\frac{16}{15}$ (C) $\frac{49}{24}$ (D) $\frac{34}{15}$

$\frac{5^{k-1}}{8^{k-1}} - \frac{3^{k-1}}{8^{k-1}} = \frac{8}{5}\sum\left(\frac{5}{8}\right)^k - \frac{8}{3}\sum\left(\frac{3}{8}\right)^k$

$\frac{8}{5}\cdot\left(\frac{\frac{5}{8}}{1-5/8}\right) - \frac{8}{3}\left(\frac{\frac{3}{8}}{1-3/8}\right)$

$\frac{8}{5}\left(\frac{5}{3}\right) - \frac{8}{3}\left(\frac{3}{5}\right)$

$\frac{8}{3} - \frac{8}{5} = \boxed{\frac{16}{15}}$

5. Determine whether the series $\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$ converges or diverges. If it converges, find bounds for the sum of the series.

(A) Converges; $\frac{3}{2} < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < \frac{5}{2}$

(B) Converges; $2 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 3$

$\boxed{\text{(C)}}$ Converges; $4 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 6$

(D) The series diverges.

* Sum of convergent

$$\frac{1}{p-1} < \sum_{i=1}^{\infty} \frac{1}{x^p} < 1 + \frac{1}{p-1}$$

$$2\left[\frac{1}{3/2 - 1} < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < 1 + \frac{1}{\frac{3}{2}-1}\right]$$

$$\boxed{4 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 6}$$

6. (a) Given the infinite series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$, find a function f with the property that $f(k) = a_k$ for all positive integers $k \geq 3$.

(b) Show that f is continuous, positive, and decreasing on the interval $[3, \infty)$.

(c) Determine whether the series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ converges or diverges.

a) $f(x) = \frac{\ln x}{x}$

b) $f'(x) = \frac{1 - \ln x}{x^2}$ $\quad$ 0 $\quad$ e

$f(x)$ is continuous, positive, decreasing on interval $[3, \infty)$

$$\int_3^{\infty} \frac{\ln x}{x}\,dx$$

$u = \ln x$ $\quad$ $dx = x\,du$

$\frac{du}{dx} = \frac{1}{x}$

$\int \frac{u}{x} \cdot x\,du \rightarrow \frac{u^2}{2}$

$$\left.\frac{(\ln x)^2}{2}\right]_3^b = \lim_{b \to \infty} \frac{(\ln b)^2}{2} - \frac{(\ln 3)^2}{2} = \infty$$

By Integral Test, the series also diverge.

7. (a) Show that the infinite series $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$ converges.

42 (b) Find bounds for the sum of the series $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$.

a) $f(x) = \frac{1}{1+9x^2}$ is positive, continuous, decreasing $[1, \infty)$

$$\int \frac{1}{1+9x^2}\,dx \rightarrow \int \frac{1}{(1)^2 + (3x)^2}\,dx \quad \frac{1}{1} \cdot \frac{1}{3} \arctan\left(\frac{3x}{1}\right)\Big]_1^b = \lim_{b \to \infty} \frac{1}{3}\arctan(b) - \frac{1}{3}\arctan(3)$$

$u = 3x$ $\quad$ $dx = \frac{du}{3}$

$du/dx = 3$

$$= \boxed{\frac{1}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3}\tan^{-1}(3)}$$

Since the Integral converge, then series also converge.