Key

10.4 AP Practice Problems (p.754) – Direct Comparison Test & Limit Comparison Test

1. Which of the following series converge?

I. $\sum_{k=1}^{\infty} \frac{2\pi^k}{3^k \pi}$ II. $\sum_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ III. $\sum_{k=1}^{\infty} \frac{k^2+3\sqrt[3]{k}}{2k^5}$

(A) I only $\boxed{\text{(B)}}$ III only

(C) I and III only (D) II and III only

iii) $\sum \frac{k^2}{2k^5} + \frac{3k^{1/3}}{2k^5}$

$\sum \frac{1}{2k^3} + \frac{3}{2k^{14/3}}$

converges by p-series test since $p=3$ and $p=14/3$ (both instances $p>1$)

i) $\sum \frac{2}{\pi}\cdot\left(\frac{\pi}{3}\right)^k$ diverges since $r=\frac{\pi}{3}>1$ (GST)

ii) $\sum \frac{k^2}{2k^3+1}$ → pos, dec, continuous

$\int_1^b \frac{x^2}{2x^3+1}dx$ | $\int \frac{x^2}{u}\cdot\frac{du}{6x^2}$ | $\frac{1}{6}\ln|u|$

$u=2x^3+1$

$du/dx=6x^2$

$\frac{1}{6}\int\frac{1}{u}du$ | $\frac{1}{6}\ln|2x^3+1|\Big]_1^b = \frac{1}{6}\ln|2b^3+1|-\frac{1}{6}\ln|3| = \infty$ $\boxed{\text{(diverges)}}$

2. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \frac{7k-5}{k^3}$ II. $\sum_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ III. $\sum_{k=1}^{\infty} \frac{k+3}{(k-3)^2+1}$

(A) II only (B) III only

(C) I and II only $\boxed{\text{(D)}}$ II and III only

i) $\frac{7k}{k^3}-\frac{5}{k^3} \to \frac{7}{k^2}-\frac{5}{k^3}$ (converges by p-series)

ii) Integral test diverges (see #1 above)

iii) $\sum\frac{k+3}{k^2-6k+10} \to \sum\frac{n+3}{n^2-6n+10}$ → limit comparison test with $\frac{1}{n}$ (diverging harmonic series)

$\lim_{n\to\infty}\frac{\frac{n+3}{n^2-6n+10}}{\frac{1}{n}} \to \lim_{n\to\infty}\frac{n+3}{n^2-6n+10}\cdot\frac{n}{1} \to \lim_{n\to\infty}\frac{n^2+3n}{n^2-6n+10} = \boxed{1}$

series also diverge by LCT

3. Determine whether the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2+2k+1}$ converges or diverges. Be sure to show your work.

Compare with $\frac{1}{k^2}$ (converging p-series)

$0 < \frac{1}{(k+1)^2} < \frac{1}{k^2}$ By DCT, series also converge.

4. Determine whether the series $\sum_{k=1}^{\infty} \frac{2k^2-1}{k(k^2+3)}$ converges or diverges. Be sure to show your work.

$\sum_{n=1}^{\infty} \frac{2n^2-1}{n^3+3n}$ → use LCT (compare with $\frac{1}{n}$, harmonic diverging series)

$$\frac{\lim_{n\to\infty} \frac{2n^2-1}{n^3+3n}}{\frac{1}{n}} \to \lim_{n\to\infty} \frac{2n^2-1}{n^3+3n} \cdot \frac{n}{1} \to \lim_{n\to\infty} \frac{2n^3-n}{n^3+3n} = 2$$

By LCT, series also diverge