10.6 AP Practice Problems (p. 772) – Ratio Test & Root Test

1. Which of the following series converge?

I. $\displaystyle\sum_{k=1}^{\infty}\frac{k^2}{(3k+1)!}$

II. $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}$

III. $\displaystyle\sum_{k=1}^{\infty}k\left(\frac{2}{3}\right)^k$

(A) II only (B) III only

(C) I and III only (D) II and III only

ii) $\lim_{n\to\infty}\frac{(k+1)^{k+1}}{(k+1)!}\cdot\frac{k!}{k^k} \to \frac{(k+1)^k\cdot\cancel{(k+1)}\cdot k!}{\cancel{(k+1)}!\,k^k}$

$\lim_{n\to\infty}\left(\frac{k+1}{k}\right)^k \to \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \to \left(\frac{n}{n}+\frac{1}{n}\right)^n$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$ (diverges)

iii) $\lim_{n\to\infty}\left(\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n}\right) \to \frac{n+1}{n}\cdot\frac{\cancel{\left(\frac{2}{3}\right)^n}\left(\frac{2}{3}\right)}{\cancel{\left(\frac{2}{3}\right)^n}}$

$= \frac{2}{3} < 1$ converges

I. Ratio Test

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$

$\lim_{n\to\infty}\frac{(k+1)^2}{(3(k+1)+1)!}\cdot\frac{(3k+1)!}{k^2}$

$\lim_{n\to\infty}\frac{(k+1)^2}{(3k+4)!}\cdot\frac{(3k+1)!}{k^2}$

$\lim_{n\to\infty}\frac{(k+1)^2(3k+1)!}{(3k+4)(3k+3)(3k+2)(3k+1)!\,k^2} = 0 < 1$

2. For which of the following series does the Ratio Test provide no information?

I. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k+1}$ **II.** $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ **III.** $\displaystyle\sum_{k=1}^{\infty}\frac{k^2+3k}{k+1}$

(A) I only (B) II only

(C) I and II only (D) I, II, and III

i) $\lim_{n\to\infty}\frac{\sqrt{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}} = 1$ (inconclusive)

ii) $\lim_{n\to\infty}\frac{1}{(n+1)\ln(n+1)}\cdot\frac{n\ln n}{1} = 1$ (inconclusive)

iii) $\lim_{k\to\infty}\frac{(k+1)^2+3(k+1)}{k+2}\cdot\frac{k+1}{k^2+3k} = 1$ (inconclusive)

3. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \frac{k!}{100^k}$

II. $\sum_{k=1}^{\infty} \frac{20^k}{2^{k^2}}$

III. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$ converges by Ratio Test or LCT

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

i) $\lim_{n\to\infty} \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} \to \frac{(n+1)\cancel{n!}(\cancel{100^n})}{\cancel{100^n}\cdot 100 \cdot \cancel{n!}}$

$= \infty > 0$ (diverges)

ii) $\lim_{n\to\infty} \frac{20^{k+1}}{2^{(k+1)^2}} \cdot \frac{2^{k^2}}{20^k} \to \frac{}{2^{k^2+2k+1}}$

$\frac{\cancel{20^k}\cdot 20 \cdot \cancel{2^{k^2}}}{\cancel{2^{k^2}}\cdot 2^k \cdot 2 \cdot \cancel{20^k}} = 0 < 1$

(Converges)

iii) $\lim_{n\to\infty} \frac{\sqrt{n}}{n^3+1} \cdot \frac{n^{5/2}}{1} = \boxed{1}$

LCT compare w/ $\frac{\sqrt{n}}{n^3} \to \frac{1}{n^{5/2}}$

4. Determine whether $\sum_{k=1}^{\infty} \frac{e^k}{k^k}$ converges or diverges.

Show your work.

$\left(\frac{e}{k}\right)^k \to \lim_{n\to\infty} \frac{e}{n} = 0 < 1$ $\boxed{\text{converges by Root test}}$

$\left(\frac{e}{n}\right)^n$

5. Show that for any p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p > 0$, the Ratio Test provides no information about whether the series converges or diverges.

$\lim_{n\to\infty} \frac{1}{(n+1)^x} \cdot \frac{n^x}{1} \to \left(\frac{n}{n+1}\right)^x = 1$ (inconclusive)