Key

10.8 AP Practice Problems (p. 788-789) – Power Series

1. If $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = -8$, then which of the following must be true?

I. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = 8$

✓ II. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = 0$

✓ III. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = -6$

(A) I only (B) II only
$\boxed{\text{(C)}}$ II and III only (D) I, II, and III

2. If the radius of convergence of the series $\sum_{k=0}^{\infty} a_k (x-3)^k$ is 2, then which of the following must be true?

I. The series converges for $x = 1$.

✓ II. The series converges for $x = 2$.

III. The series converges for $x = 5$.

(A) I only $\boxed{\text{(B)}}$ II only
(C) I and III only (D) I, II, and III

$|x-3| < 2$

$-2 < x-3 < 2$

$1 < x < 5$

*series may or may not converge at endpoints $x=1$ or $x=5$

3. The interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x+3)^k}{k}$ is

(A) $(-4, -2)$ (B) $[-4, -2)$
(C) $[-4, -2]$ (D) $x = -3$ only

*Ratio Test $\rightarrow \sum_{n=1} \frac{(x+3)^n}{n}$

$\lim_{n\to\infty} \frac{(x+3)^{n+1}}{n+1} \cdot \frac{n}{(x+3)^n}$

$\lim_{n\to\infty} \frac{(x+3)^n(x+3)\,n}{n+1\,(x+3)^n} \rightarrow |x+3| < 1$

$-1 < x+3 < 1$
$-4 < x < -2$
or
$-4 \le x \le -2$

$\frac{(-4+3)^n}{n} = \frac{(-1)^n}{n}$ converges at $x=-4$

$\frac{(-2+3)^n}{n} \rightarrow \frac{(1)^n}{n}$ diverges at $x=-2$

$\boxed{[-4, -2)}$

4. Which series has an interval of convergence $[-1, 1]$?

I. $\sum_{k=1}^{\infty} kx^k$ II. $\sum_{k=1}^{\infty} \frac{x^k}{k}$ III. $\sum_{k=1}^{\infty} \frac{x^k}{k^3}$

(A) I only (B) III only
(C) I and III only (D) II and III only

i) $\lim_{n\to\infty} \frac{(n+1)x^{n+1}}{nx^n} \rightarrow |x| < 1 \rightarrow (-1, 1)$

ii) $\lim_{n\to\infty} \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = |x| < 1 \rightarrow [-1, 1)$

iii) $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{x^n} \rightarrow |x| < 1$

$\boxed{[-1, 1]}$

5. Find all numbers x for which the power series $\sum_{k=0}^{\infty} \frac{x^k}{3^k}$ converges.

(A) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (B) $[-1, 1]$ (C) $(-3, 3)$ (D) $[-3, 3]$

$\lim_{n\to\infty} \frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n} \rightarrow \left|\frac{x}{3}\right| < 1$ $-3 < x < 3$ or $-3 \le x \le 3$

test $x=-3 \rightarrow \sum \frac{(-3)^k}{3^k} \rightarrow$ converges $\frac{3^k \cdot (-1)^k}{3^k} \rightarrow \frac{(-1)^k}{1^k}$ diverges by GST

test $x=3$ $\sum \frac{(3)^k}{3^k} \rightarrow (1)^k$ diverges by nth term test

$\boxed{(-3, 3)}$

6. The power series representation of a function f is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Then the power series for $f'(x)$ is

(A) $\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$ (B) $\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$

(C) $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k+1)!}$ (D) $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k-1)!}$

$f'(x) = \sum_{k=1} \frac{k \cdot x^{k-1}}{k!} \rightarrow \frac{\cancel{k} \cdot x^{k-1}}{\cancel{k} \cdot (k-1)!} \rightarrow \boxed{\frac{x^{k-1}}{(k-1)!}} \rightarrow$

$$\boxed{f'(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}}$$

$f(x) = \frac{(x-2)^n}{3^n}$

7. (a) Find the radius of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k}$$

(b) What is the interval of convergence for the power series?

$|x-c| < r$

a) $\lim_{n\to\infty} \frac{(x-2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-2)^n} \rightarrow \lim_{n\to\infty} \left|\frac{(x-2)}{3}\right| < 1 \rightarrow |x-2| < 3$

$\boxed{r=3}$

b) IOC is $-3 < x-2 < 3$

$-1 < x < 5$

test $x=-1$: $\sum \frac{(-1-2)^n}{3^n} \rightarrow \frac{(-3)^n}{3^n} \rightarrow$ diverges by AST

$\boxed{(-1, 5)}$

test $x=5$: $\sum \frac{(5-2)^k}{3^k} = 1 \rightarrow$ diverges by n^{th} term test

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8. (a) Use the power series

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}, -1 < x \leq 1,$$ to find the power series representation for $f(x) = \ln(1+x^2)$.

(b) Use properties of power series to find the derivative of f.

(c) What is the interval of convergence for f'?

$$i)\ f(x) = \ln(1+x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{k+1}}{k+1} \rightarrow \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$$

$$b)\ f'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (2k+2)\, x^{2k+1}}{k+1} \rightarrow 2\sum_{k=0}^{\infty} (-1)^k \frac{k+1 \cdot x^{2k+1}}{k+1}$$

$$2\sum_{k=0}^{\infty} (-1)^k x^{2k+1} = 2x - 2x^3 + 2x^5 - 2x^7 + \ldots (-1)^k(2x^{2k+1}) + \ldots$$

$$c)\ \lim_{n\to\infty} \frac{2x^{2(k+1)+1}}{2x^{2k+1}} \rightarrow \frac{x^{2k+3}}{x^{2k+1}} \rightarrow |x^2| < 1 \qquad -1 < x < 1$$

test $x=-1$: $(-1)^k(-1)^{2k+1} \rightarrow (-1)^{3k+1} = -1+1-1+1 \rightarrow$ diverging

test $x=1$: $(-1)^k(1)^{2k+1} \rightarrow +1-1+1-1$ diverges

$$\boxed{\text{I.O.C} \rightarrow (-1,1)}$$