Key

10.9 AP Practice Problems (p. 799) – Taylor Series

1. The coefficient of x^8 in the Maclaurin expansion for $f(x) = (3x)^2 \cos x$ is

(A) $-\frac{1}{240}$ (B) $-\frac{1}{80}$ (C) $\frac{1}{80}$ (D) $\frac{3}{8}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \to \frac{(-1)^n x^{2n}}{(2n)!}$

$(3x)^2 \cos x \to 9x^2\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\right] \to \frac{(-1)^n \cdot 9x^{2n+2}}{(2n)!}$

$\frac{-9x^2 \cdot x^6}{6!} \to \frac{-9x^8}{6!}$

Coefficient is $\frac{-9}{6!} = \boxed{-\frac{1}{80}}$

2. The Maclaurin expansion for $f(x) = e^{x/3}$ is

(A) $\sum_{k=0}^{\infty} \frac{x^k}{3^k}$ (B) $\sum_{k=0}^{\infty} \frac{x^k}{3k!}$

(C) $\sum_{k=0}^{\infty} \frac{x^k}{3^k k!}$ (D) $\sum_{k=0}^{\infty} \frac{x^k}{(3k)!}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$e^{x/3} = 1 + \frac{x}{3} + \frac{(x/3)^2}{2!} + \frac{(x/3)^3}{3!} + \cdots \sum \frac{(x/3)^k}{k!} \to \left(\frac{x}{3}\right)^k \cdot \frac{1}{k!} \to \boxed{\sum_{k=0}^{\infty} \frac{x^k}{3^k \cdot k!}}$

3. The Maclaurin expansion of $f(x) = \frac{e^{2x} - 1}{x}$ is

(A) $\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$ (B) $\sum_{k=0}^{\infty} \frac{2^{k+1} x^k}{(k+1)!}$

(C) $\sum_{k=0}^{\infty} \frac{(2x)^k}{(k+1)!}$ (D) $\sum_{k=0}^{\infty} \frac{2x^k}{(k+1)!}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \to \sum \frac{x^k}{k!}$

$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}$

$e^{2x} - 1 = (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}$

$\frac{1}{x}(e^{2x} - 1) = \frac{1}{x}\left[(2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right]$

$\sum \frac{(2x)^k}{k! \cdot x} \to \frac{2^k x^k}{k! \cdot x^1} \to \frac{2^k \cdot x^{k-1}}{k!}$

$\sum_{k=1}^{\infty} \frac{2^k x^{k-1}}{k!} \to$ or $\to \boxed{\sum_{k=0}^{\infty} \frac{2^{k+1} x^k}{(k+1)!}}$

4. The Taylor expansion for $\int_0^x \cos t\, dt$ about $\frac{\pi}{2}$ is

(A) $\sum_{k=0}^{\infty}(-1)^k \frac{\left(x-\frac{\pi}{2}\right)^{2k}}{(2k)!}$ (B) $\sum_{k=0}^{\infty}(-1)^k \frac{\left(x-\frac{\pi}{2}\right)^{2k+1}}{2k+1}$

(C) $\sum_{k=0}^{\infty}(-1)^k \frac{\left(x-\frac{\pi}{2}\right)^{k}}{k!}$ (D) $\sum_{k=0}^{\infty}(-1)^k \frac{\left(x-\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \frac{(-1)^n t^{2n}}{(2n)!}$$

$$\int \cos(t)\,dt = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \ldots \frac{(-1)^n t^{2n+1}}{(2n+1)(2n)!} \to \left.\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right]_0^x$$

$$\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

centered about $c = \pi/2$

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k (x-\pi/2)^{2k+1}}{(2k+1)!}}$$

5. What is the first nonzero term of the Maclaurin expansion of $f(x) = \ln(2x^3+1)$?

(A) $\ln(3x^2+1)$ (B) $6x^2$ (C) $2x^3$ (D) $3x^2$

$$\frac{1}{x+1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots (-1)^n x^n$$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots \frac{(-1)^n x^{n+1}}{n+1}$$

$$\ln(2x^3+1) = (2x^3) - \frac{(2x^3)^2}{2} + \frac{(2x^3)^3}{3} - \frac{(2x^3)^4}{4} + \ldots \frac{(-1)^n (2x^3)^{n+1}}{n+1}$$

$\boxed{\text{First nonzero term is } 2x^3}$

6. The Maclaurin expansion of $f(x) = \frac{1}{(1-x)^2}$ equals $\sum_{k=0}^{\infty} a_k x^k$.

(a) Find the coefficients of the first four terms of the Maclaurin expansion.

(b) Given the Maclaurin expansion of $g(x) = \frac{2}{(1-x)^3}$ $\rightarrow 2(1-x)^{-3}$

equals $\sum_{k=0}^{\infty} b_k x^k$, express b_k in terms of a_k.

a) $\sum_{k=0}^{\infty} a_k x^k = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$ $\boxed{\text{coefficients are } 1, 2, 3, \text{ and } 4}$

b) $f(0) = 2$

$f'(x) = -6((1-x)^{-4}(-1)) \rightarrow f'(0) = +6$

$f''(x) = -24(1-x)^{-5}(-1) \rightarrow f''(0) = -24$

$f'''(0) = -120(1-x)^{-6}(-1) \rightarrow f'''(0) = 120$

Maclaurin Expansion: $\frac{2}{(1-x)^3} \rightarrow f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$\rightarrow 2 + 6x + \frac{24x^2}{2!} + \frac{120x^3}{3!}$

$\rightarrow 2 + 6x + 12x^2 + 20x^3 + \ldots$

$\sum_{k=0}^{\infty} b_k x^k = \boxed{b_k = a_k(k+2)}$