Name: ______________________________ Period: __________

BC Calculus

Unit 10 Packet
Infinite Series
(Part 1)

Tests for Convergence

(n^{th} term test, Geometric series test, p-series, Integral test, Direct & Limit Comparison tests, Ratio test, & Root test)

BC Calculus – 10.1 Notes – Convergent & Divergent Infinite Series

Recall: Writing terms of a sequence.

$$a_n = \{1 + (-2)^n\}$$

$$-1, 5, -7, 17, -31$$

Sequence: A collection of numbers that are in one-to-one correspondence with positive integers.

$$-2 \qquad 4 \qquad -\frac{26}{6} \qquad \frac{80}{24} \qquad -\frac{242}{120}$$

Monotonic Sequences never decreases or never increases	**Bounded Sequences**
$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$ or $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n$	$a_n \leq M$ (upper bound / above) $a_n \geq N$ (lower bound / below) $\{a_n\}$ bounded if both are true

Infinite Series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Partial Sum:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

a_n vs S_n:

a_n is an expression that gives the

S_n is an expression that gives the

1. Use the following sequence $2, 4, 6, 8, 10$ to find a_4 and S_4.

$$\sum_{n=1}^{\infty} a_n =$$

②

Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the n^{th} partial sum is $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

If the sequence of the partial sum $\{S_n\}$ to S, then the series $\sum_{n=1}^{\infty} a_n$ The limit S is called the sum of the series.

Likewise, if $\{S_n\}$ then the series

2. **Does the series converge or diverge?** $\sum_{n=1}^{\infty} \frac{1}{2^n}$

3. Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{10}{n(n+2)}$ for $n = 200, 1000$.

4. **Does the series converge or diverge?** $\sum_{n=1}^{\infty} n$

10.1 Convergent and Divergent Infinite Series

Calculus

Practice

1. Given the infinite series $\sum_{n=1}^{\infty} (-1)^n$, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.

2. Find the sequence of partial sums $S_1, S_2, S_3, S_4,$ and S_5 for the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$.

3. If the infinite series $\sum_{n=1}^{\infty} a^n$ has nth partial sum $S_n = (-1)^{n+1}$ for $n \geq 1$, what is the sum of the series?

4. The infinite series $\sum_{n=1}^{\infty} a^n$ has nth partial sum $S_n = \frac{n}{4n+1}$ for $n \geq 1$. What is the sum of the series?

5. Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$ for $n = 100, 500, 1000$.

6. Show that the sequence with the given nth term $a_n = 1 + 2n$ is monotonic.

7. What is the nth partial sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$?

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8. Which of the following could be the nth partial sum for the infinite series $\sum_{n=1}^{\infty} \frac{1}{4^n}$?

(A) $S_n = \frac{1}{3}\left(1 + \frac{1}{4^n}\right)$ (B) $S_n = \frac{1}{3}\left(1 - \frac{1}{4^{n+1}}\right)$ (C) $S_n = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$ (D) $S_n = \frac{1}{4}\left(1 - \frac{1}{3^n}\right)$

9. If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and has a sum of $\frac{7}{8}$, which of the following could be the nth partial sum?

(A) $S_n = \frac{7n+1}{8n^2+1}$

(B) $S_n = \frac{7n^2+1}{8n+1}$

(C) $S_n = 2\left(\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3}\right)$

(D) $S_n = \left(\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3}\right)$

10. Which of the following sequences with the given nth term is bounded and monotonic?

(A) $a_n = 2 + (-1)^n$ (B) $a_n = \frac{n^2}{n+1}$ (C) $a_n = \frac{3n}{n+2}$ (D) $a_n = \frac{\cos n}{n}$

10.1 AP Practice Problems (p.721) - Sequences

1. The general term a_n for the sequence

$$\left\{0, -\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, \ldots\right\} \text{ is}$$

(A) $\frac{(n-1)^2}{n}$

(B) $(-1)^n \frac{(n-1)^2}{n}$

(C) $(-1)^{n+1} \frac{(n-1)^2}{n}$

(D) $(-1)^n \frac{n^2}{n+1}$

2. Which sequence is defined by $\{b_n\} = \left\{\left(-\frac{2}{3}\right)^n (n-3)\right\}$?

(A) $\frac{4}{3}, -\frac{4}{9}, \frac{16}{81}, -\frac{64}{243}, \ldots$

(B) $\frac{4}{3}, -\frac{4}{9}, 0, \frac{16}{81}, -\frac{64}{243}, \ldots$

(C) $-\frac{2}{3}, \frac{4}{9}, 0, \frac{16}{81}, -\frac{32}{243}, \ldots$

(D) $-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, -\frac{32}{243}, \ldots$

3. The sequence $\{a_n\} = \left\{\frac{2}{3n+1}\right\}$ can be described as

(A) increasing and bounded.

(B) decreasing and bounded.

(C) increasing and not bounded.

(D) decreasing and not bounded.

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4. For what values of x does the sequence $\{3x^n\}$ converge?

(A) $-1 < x \leq 1$ (B) $-1 < x < 1$

(C) $-3 < x < 3$ (D) The sequence diverges.

5. Determine whether the sequence $\left\{\dfrac{\sin n}{n^3}\right\}$ converges or diverges. Justify your answer.

BC Calculus – 10.2 Notes – Geometric Series

Recall: What is a geometric sequence?
A geometric sequence is one in which the same number is to each term to get the next term in the sequence. The number you multiply by is called the usually denoted by

n^{th} Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is given by:

$a_n =$ or $a_n =$ or $a_n =$

1. $3, 6, 12, 24, 48, \ldots$

$a_n = a_0 r^n$	$a_n = a_1 r^{n-1}$	$a_n = a_2 r^{n-2}$
$a_n =$	$a_n =$	$a_n =$

2. $25, 5, 1, \frac{1}{5}, \frac{1}{25}, \ldots$

$a_n = a_0 r^n$	$a_n = a_1 r^{n-1}$	$a_n = a_2 r^{n-2}$
$a_n =$	$a_n =$	$a_n =$

$$\sum_{n=0}^{\infty} ar^n =$$

$a \neq 0$

Geometric Infinite Series Convergence

A geometric series with ratio r diverges when If then the series converges to

$$\sum_{n=k}^{\infty} ar^n =$$

Where is the first term of the series.

1. $\sum_{n=0}^{\infty} \frac{3}{4^n}$

2. $\sum_{n=2}^{\infty} \frac{3^{n+1}}{4^n}$

(8)

3. For what value of r does the infinite series $\sum_{n=0}^{\infty} 17r^n$ equal 23?

4. **Calculator active.** If $f(x) = \sum_{n=3}^{\infty} \left(\sin^2\left(\frac{x}{3}\right)\right)^n$, then $f(7) =$

<u>10.2 Practice problems:</u>

Find the value of each infinite series.	
1. $\sum_{n=1}^{\infty} -\frac{7}{(-3)^n}$	2. $\sum_{n=0}^{\infty} \frac{1}{3^n}$
3. $\sum_{n=0}^{\infty} e^{nx}$ Let x be a real number, with $x < 0$.	4. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

5. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$

6. $\sum_{n=1}^{\infty} \frac{2^n}{e^{n+1}}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi}{e^{n+1}}$

8. $\sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$

9. What is the sum of the infinite series
$25 + -5 + 1 + -\frac{1}{5} + \frac{1}{25} + \cdots$

10. **Calculator active.** If $f(x) = \sum_{n=1}^{\infty} (\sin^2 2x)^n$, then $f(3) =$

(10)

11. For what value of a does the infinite series

$$\sum_{n=0}^{\infty} a\left(\frac{2}{3}\right)^n = 14$$

12. Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$.

The first term of the series $a_1 = 24$, and the third term $a_3 = 6$. What are possible values for a_2?

13. Consider the series $\sum_{n=1}^{\infty} a_n$. If $a_1 = 32$ and $\frac{a_{n+1}}{a_n} = \frac{1}{4}$ for all integers $n \geq 1$, then $\sum_{n=1}^{\infty} a_n =$

14. Use a geometric series to write $0.\overline{2}$ as the ratio of two integers.

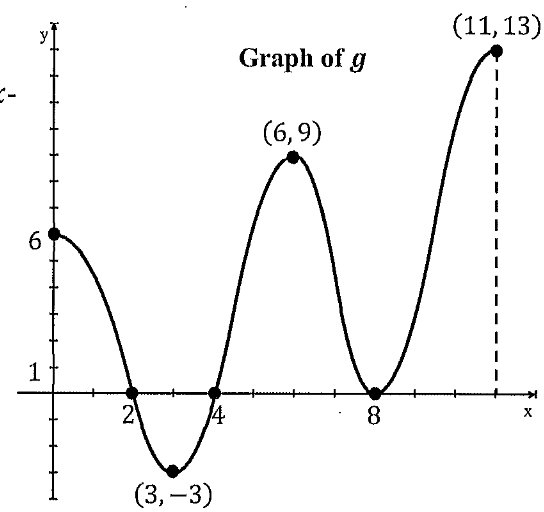
10.2 Working with Geometric Series

Test Prep

15. If x and y are positive real numbers, which of the following conditions guarantees the infinite series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{y^{2n+1}}$ is geometric and converges?

(A) $x < y$ (B) $x < y^2$ (C) $x > y^2$ (D) $x > y$

16. The figure to the right shows a portion of the graph of the differentiable function g. Let h be the function defined by $h(x) = \int_4^x g(t)\,dt$. The areas of the regions bounded by the x-axis and the graph of g on the intervals, $[0,2]$, $[2,4]$, $[4,8]$ and $[8,11]$ are $6, 4, 24$, and 19, respectively.



a. Must there exist a value of c, for $2 < c < 4$, such that $h(c) = 3.5$? Justify your answer.

b. Find the average value of g over the interval, $0 \le x \le 11$. Show the computations that lead to your answer.

c. Evaluate $\lim\limits_{x \to 8} \frac{h(x) - 3x}{x^2 - 64}$.

d. Is there a value r such that the series $30 + 30r + 30r^2 + \cdots + 30r^n$ equals the value of $g(6)$?

(12)

10.2 AP Practice Problems (p.734) – Infinite Series & Tests for Convergence (Geometric Series Test)

1. Find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$, if it exists.

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) The series diverges.

2. $\sum_{k=1}^{\infty} \frac{7}{3^{k-1}}$

(A) converges and equals $\frac{7}{4}$.

(B) converges and equals $\frac{14}{3}$.

(C) converges and equals $\frac{21}{2}$.

(D) diverges.

3. The repeating decimal $0.1212\ldots$ can be expressed as the fraction

(A) $\frac{4}{33}$ (B) $\frac{3}{25}$ (C) $\frac{4}{99}$ (D) $\frac{303}{250}$

4. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \left(\sqrt{2}\right)^{k-1}$ II. $\sum_{k=1}^{\infty} -\frac{3}{4^k}$ III. $\sum_{k=1}^{\infty} \frac{1}{k}$

(A) I and II only (B) I and III only

(C) II and III only (D) I, II, and III

5. Determine whether the series $\sum_{k=1}^{\infty} \frac{7^{k-2}}{8^{k+1}}$ converges or diverges. If it converges, find its sum.

(A) converges and equals $\frac{1}{64}$

(B) converges and equals $\frac{1}{56}$

(C) converges and equals $\frac{1}{8}$

(D) The series diverges.

6. An object hanging on a spring is pulled downward a distance of 100 cm from its equilibrium position (the origin) and released. It recoils upward past the origin to a height 90 cm above the origin. It continues to oscillate up and down about the origin, but with each oscillation the spring travels only $\frac{9}{10}$ of the last distance traveled.

(a) Write an infinite series that models the object's movement.

(b) Find the sum of the infinite series.

(c) Interpret the sum of the infinite series in the context of the problem.

(14)

BC Calculus – 10.3 Notes – *n*th term, p-series, and Integral Test

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n =$

If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$

Nth Term Test for Divergence (10.3a)

If $\lim_{n\to\infty} a_n \neq 0$, then

Use the Nth term test to make a conclusion about divergence for each series.

1. $\sum_{n=1}^{\infty} \frac{3n^3+1}{5n^3-2n^2+1}$

2. $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$

3. $\sum_{n=1}^{\infty} \frac{1}{n}$

4. $\sum_{n=1}^{\infty} \frac{2^{n+2}}{2^{n+3}+1}$

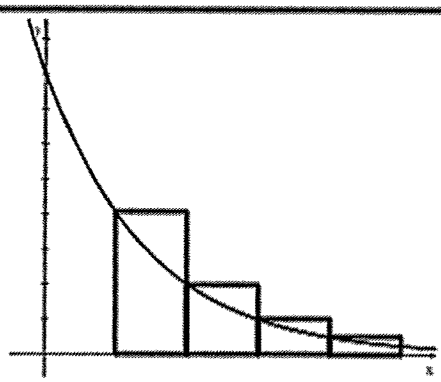
5. $\sum_{n=1}^{\infty} \frac{e^{4n}}{3n}$

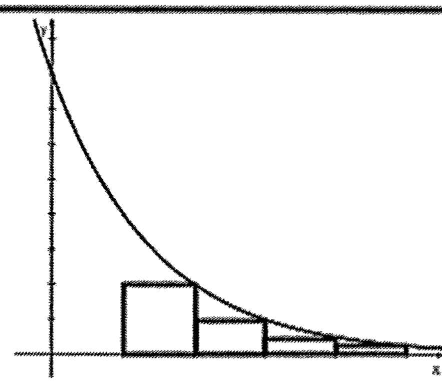
10.3b

Integral Test for Convergence

If f is a positive, continuous, and decreasing function for $x \geq k$, and $a_n = f(x)$, then

and





Determine the convergence or divergence of the series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

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10.3c

p-Series

Let p be a positive constant of the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

The series converges if

The series diverges if

Harmonic Series

Do the following series converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

For what values of k will the series converge?

3. $\sum_{n=1}^{\infty} \frac{1}{n^{2k-5}}$

4. $\sum_{n=1}^{\infty} \frac{1}{n(n^{2k})}$

5. $\sum_{n=1}^{\infty} \frac{n}{n^{4k}+5}$

Things we should now recognize

Series

- Geometric
- Harmonic
- p-Series

Tests for convergence/divergence

- Nth Term Test for Divergence
- Integral Test

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10.3 The nth Term Test for Divergence (10.3a)

Calculus

Practice

For each of the following series, determine the convergence or divergence of the given series. State the reasoning behind your answer.

1. $\sum_{n=1}^{\infty} \frac{3-2n}{5n+1}$

2. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$

3. $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^2+1}}$

4. $\sum_{n=1}^{\infty} \frac{e^{n+1}}{\pi^n}$

5. $\sum_{n=1}^{\infty} \frac{7^n+1}{7^{n+1}}$

6. $\sum_{n=0}^{\infty} 5\left(\frac{5}{2}\right)^n$

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10.3 The nth Term Test for Divergence (10.3a)

Test Prep

7. The nth-Term Test can be used to determine divergence for which of the following series?

I. $\sum_{n=1}^{\infty} \sin 2n$

II. $\sum_{n=1}^{\infty} \left(2 + \frac{3}{n}\right)$

III. $\sum_{n=1}^{\infty} \frac{n^3 + 1}{n^2}$

(A) II only

(B) III only

(C) I and II only

(D) I, II, and III

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8. The nth-Term Test can be used to determine divergence for which of the following series?

I. $\sum_{n=1}^{\infty} \ln\left(\frac{n-1}{n}\right)$

II. $\sum_{n=1}^{\infty} \frac{3n - 2n^2}{5n^2}$

III. $\sum_{n=1}^{\infty} 3\left(\frac{5}{4}\right)^n$

(A) II only

(B) II and III only

(C) I and II only

(D) I, II, and III

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9. If $a_n = \cos\left(\frac{\pi}{2n}\right)$ for $n = 1,2,3,\ldots$, which of the following about $\sum_{n=1}^{\infty} a_n$ must be true?

(A) The series converges and $\lim_{n\to\infty} a_n = 0$.

(B) The series diverges and $\lim_{n\to\infty} a_n = 0$

(C) The series diverges and $\lim_{n\to\infty} a_n \neq 0$

(D) The series converges and $\lim_{n\to\infty} a_n \neq 0$

10.3b Integral Test for Convergence

Calculus

Practice

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If the Integral Test applies, use it to determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

2. $\sum_{n=1}^{\infty} f(n) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

3. $\sum_{n=1}^{\infty} f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

4. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{n}$

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5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+2}$, where k is a positive integer. Assume the series meets the criteria for the Integral Test.

7. Let f be a positive, continuous, and decreasing function. If $\int_1^{\infty} f(x)\,dx = 4$, which of the following statements about the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ must be true?

A. $\displaystyle\sum_{n=1}^{\infty} f(n) = 0$

B. $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges, and $\displaystyle\sum_{n=1}^{\infty} f(n) > 4$

C. $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges, and $\displaystyle\sum_{n=1}^{\infty} f(n) < 4$

D. $\displaystyle\sum_{n=1}^{\infty} f(n)$ diverges, and $\displaystyle\sum_{n=1}^{\infty} f(n) = 0$

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8. Explain why the Integral Test does not apply for the series $\displaystyle\sum_{x=1}^{\infty} e^x \sin x$.

9. Show that the series $\displaystyle\sum_{x=1}^{\infty} \frac{\tan^{-1} x}{x^2+1}$ meets the criteria to apply the Integral Test for convergence.

10. Let f be positive, continuous, and decreasing on $[1, \infty)$, such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n = 7$, which of the following must be true?

A. $\lim_{n\to\infty} a_n = 7$

B. $\int_1^{\infty} f(x)\,dx = 7$

C. $\int_1^{\infty} f(x)\,dx$ diverges

D. $\int_1^{\infty} f(x)\,dx$ converges

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11. Which of the following can be used to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n+3}{n+4}$?

I. Properties of Geometric Series
II. nth-Term Test
III. Integral Test

A. I only

B. II only

C. III only

D. II and III only

E. I, II, and III

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12. Which of the following can be used to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

I. Properties of Geometric Series
II. nth-Term Test
III. Integral Test

A. I only

B. II only

C. III only

D. I and II only

E. I and III only

10.3b Integral Test for Convergence

Test Prep

13. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. The integral test can be used to determine convergence or divergence of the series because $f(x) = \frac{1}{x^3}$ is positive, continuous, and decreasing on $[1, \infty)$. Which of the following is true?

A. $1 + \int_1^{\infty} \frac{1}{x^3}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}\,dx$

B. $\int_1^{\infty} \frac{1}{x^3}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^3} < 1 + \int_1^{\infty} \frac{1}{x^3}\,dx$

C. $\sum_{n=1}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}\,dx < 1 + \int_1^{\infty} \frac{1}{x^3}\,dx$

D. $\int_1^{\infty} \frac{1}{x^3}\,dx < 1 + \int_1^{\infty} \frac{1}{x^3}\,dx < \sum_{n=1}^{\infty} \frac{1}{n^3}$

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10.3c Harmonic Series and p-series

Calculus

Practice

Determine the convergence or divergence of the following p-series.

1. $\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.13}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

What are all the values of p for which…

4. $\sum_{n=1}^{\infty} \frac{2n}{n^p + 2}$ converges?

5. $\sum_{n=1}^{\infty} \frac{1}{n^{3p}}$ diverges?

6. Both series $\sum_{n=1}^{\infty} n^{-5p}$ and $\sum_{n=1}^{\infty} \left(\frac{p}{5}\right)^n$ converge?

7. $\int_1^{\infty} \frac{1}{x^{3p+4}}\, dx$ converges?

Find the positive values of p for which the infinite series converge?

8. $\sum_{n=1}^{\infty} \left(\frac{4}{p}\right)^n$

9. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^p}$

10. $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$

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10.3c Harmonic Series and p-series

Test Prep

11. Which of the following infinite series converge?

I. $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$　　II. $\sum_{n=1}^{\infty} \left(\frac{e}{2}\right)^{-n}$　　III. $\sum_{n=1}^{\infty} \frac{1}{n^{e}}$

A. None　　B. II only　　C. III only

D. I and II only　　E. II and III only

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12. Which of the following infinite series converge?

I. $\sum_{n=1}^{\infty} 3^{-n}$　　II. $\sum_{n=1}^{\infty} \frac{1}{(3n+1)^3}$　　III. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

A. I only　　B. II only　　C. III only

D. I and II only　　E. I and III only

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13. Which of the following infinite series is a divergent p-series?

A. $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n}$　　B. $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$　　C. $\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$　　D. $\sum_{n=1}^{\infty} n^{\frac{3}{2}}$

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14. Which of the following is not a p-series?

A. $\sum_{n=1}^{\infty} n^{-3}$ B. $\sum_{n=1}^{\infty} \frac{1}{n}$ C. $\sum_{n=1}^{25} \frac{1}{n^{\pi}}$ D. $\sum_{n=1}^{\infty} \frac{1}{\pi^{n}}$

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15. Which of the following is a harmonic series?

A. $\sum_{n=1}^{\infty} \frac{1}{3n}$ B. $\sum_{n=1}^{\infty} \frac{1}{n}$ C. $\sum_{n=1}^{1000} \frac{1}{n}$ D. $\sum_{n=1}^{\infty} \frac{3n^2}{4n^2+1}$

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16. Find the positive values of k for which the series $\sum_{n=3}^{\infty} \frac{1}{(n \ln n)(\ln(\ln n))^k}$ converges.

10.3 AP Practice Problems (p. 746) - Tests for Convergence (nth term, p-series, Integral Test)

1. For what numbers p does the series $\sum_{k=1}^{\infty} \frac{1}{k^{p/3}}$ converge?

(A) $p > 1$ (B) $|p| > 1$ (C) $p > 3$ (D) $p > \frac{1}{3}$

2. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \frac{e^{k-1}}{3^{k-1}}$ II. $\sum_{k=1}^{\infty} \cos\left(\pi + \frac{1}{k}\right)$ III. $\sum_{k=1}^{\infty} \frac{10}{k}$

(A) I and II only (B) I and III only

(C) II and III only (D) I, II, and III

3. For what values of p does the series $\sum_{k=1}^{\infty} k^p$ converge?

(A) $-1 < p < 1$ (B) $p < -1$

(C) $p > 1$ (D) The series diverges.

4. $\sum_{k=1}^{\infty} \frac{5^{k-1} - 3^{k-1}}{8^{k-1}} =$

(A) $\frac{1}{3}$ (B) $\frac{16}{15}$ (C) $\frac{49}{24}$ (D) $\frac{34}{15}$

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5. Determine whether the series $\sum_{k=1}^{\infty} \frac{2}{k^{3/2}}$ converges or diverges. If it converges, find bounds for the sum of the series.

(A) Converges; $\frac{3}{2} < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < \frac{5}{2}$

(B) Converges; $2 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 3$

(C) Converges; $4 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 6$

(D) The series diverges.

6. (a) Given the infinite series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$, find a function f with the property that $f(k) = a_k$ for all positive integers $k \geq 3$.

(b) Show that f is continuous, positive, and decreasing on the interval $[3, \infty)$.

(c) Determine whether the series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ converges or diverges.

7. (a) Show that the infinite series $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$ converges.

(b) Find bounds for the sum of the series $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$.

BC Calculus – 10.4 Notes – Comparison Tests for Convergence

Comparison Test

Let $0 < a_n \leq b_n$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{3+2^n}$

2. $\sum_{n=1}^{\infty} \frac{1}{4^n-3}$

3. $\sum_{n=1}^{\infty} \frac{1}{7n^2+4}$

Limit Comparison Test

If $a_n > 0$, $b_n > 0$ and $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ (where L is finite and positive), then

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

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Determine if the following series converge or diverge.

4. $\sum_{n=1}^{\infty} \frac{2n^2 - 2}{5n^5 + 3n + 1}$

5. $\sum_{n=1}^{\infty} \frac{1}{5n^2 + 5n + 5}$

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+2}}$

7. $\sum_{n=1}^{\infty} \frac{n^3 - 7}{2n^5 + n^2 + n + 1}$

8. $\sum_{n=1}^{\infty} \frac{n3^n}{4n^3 + 2}$

9. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + n}}$

10.4 Comparison Tests for Convergence

Calculus

Practice

1. Which of the following statements about convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ is true?

(A) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(C) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(D) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

---

2. Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{3n}{n^3+2}$

(B) $\sum_{n=1}^{\infty} \frac{5n}{2n+1}$

(C) $\sum_{n=1}^{\infty} \frac{7n}{n^2+1}$

(D) $\sum_{n=1}^{\infty} \frac{5^n}{4^n+1}$

---

3. Use the Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges or diverges.

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4. Which of the following series can be used with the Limit Comparison Test to determine convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+3}$?

(A) $\sum_{n=1}^{\infty} \frac{n}{n+3}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n}$

(D) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

---

5. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n$ diverges which of the following must be true?

(A) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(B) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(C) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

---

6. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges which of the following must be true?

(A) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(B) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

(C) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

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7. Let $a > 0, b > 0$, and $c > 0$. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{an^2 + bn + c}$ converges or diverges.

8. Determine the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{1}{6^n + 6}$.

9. For the series $\sum_{n=1}^{\infty} \frac{n3^n}{2n^4 - 2}$, which of the following could be used with the Limit Comparison Test?

(A) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(B) $\sum_{n=1}^{\infty} \frac{3^n}{n^4}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(D) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

---

10. Which of the following can be used with the Comparison Test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$?

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(D) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

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11. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ II. $\sum_{n=1}^{\infty} \frac{1}{n^3 - 27}$ III. $\sum_{n=1}^{\infty} \frac{1}{4^n + 1}$

(A) I only

(B) II only

(C) I and II only

(D) I, II, and III

---

12. Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$, where $p \geq 0$. For what values of p is the series convergent?

13. Determine whether the series $\sum_{n=1}^{\infty} \frac{n-3}{n^3}$ converges or diverges.

14. Consider the series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-3}$. Use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{4n}$ to determine the convergence of the series.

10.4 Comparison Tests for Convergence

Test Prep

15. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $a_n \leq b_n$, then which of the following must be true?

(A) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(C) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(D) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

---

16. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$, then which of the following must be true?

I. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

II. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

III. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

IV. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(A) I only (B) II only (C) III only

(D) IV only (E) I and II only (F) III and IV only

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10.4 AP Practice Problems (p.754) – Direct Comparison Test & Limit Comparison Test

1. Which of the following series converge?

   I. $\sum_{k=1}^{\infty} \frac{2\pi^k}{3^k \pi}$ II. $\sum_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ III. $\sum_{k=1}^{\infty} \frac{k^2 + 3\sqrt[3]{k}}{2k^5}$

   (A) I only (B) III only
   (C) I and III only (D) II and III only

2. Which of the following series diverge?

   I. $\sum_{k=1}^{\infty} \frac{7k-5}{k^3}$ II. $\sum_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ III. $\sum_{k=1}^{\infty} \frac{k+3}{(k-3)^2+1}$

   (A) II only (B) III only
   (C) I and II only (D) II and III only

3. Determine whether the series $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2+2k+1}$ converges or diverges. Be sure to show your work.

4. Determine whether the series $\sum_{k=1}^{\infty} \frac{2k^2-1}{k(k^2+3)}$ converges or diverges. Be sure to show your work.

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BC Calculus – 10.5 Notes – Alternating Series Test & Absolute Convergence

Alternating Series Test

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if **BOTH** of the following conditions are met:

1.

2.

Ways to check if a_n is decreasing.

- Take the 1st derivative and see if it is negative.
- Usually, it is obvious.
- Could manipulate $a_{n+1} \le a_n$

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n+5}{(n+2)(n+3)}$

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+7)}{n}$

4. $\sum_{n=1}^{\infty}\cos(n\pi)\frac{1}{n}$

5. The following is not an alternating series. Look carefully to see if you can tell why not.

$$\sum_{n=1}^{\infty}\frac{(-1)^n \cos(n\pi)\, n}{n^2+1}$$

Absolute or Conditional Convergence Notes:

Three possibilities with regards to the series $\sum_{n=1}^{\infty} a_n$ dealing with convergence or divergence.

1. **Converges Absolutely.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then the original series $\sum_{n=1}^{\infty} a_n$ also converges.
2. **Converges Conditionally.** If $\sum_{n=1}^{\infty} |a_n|$ diverges, but the original series $\sum_{n=1}^{\infty} a_n$ converges.
3. **Divergent.** Both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ diverge.

Find if the series converges absolutely, converges conditionally, or is divergent.

1. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

Find the values of x that make the series converge absolutely.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n(x+4)^n}{6^n}$

5. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

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Write your questions and thoughts here!

5. The following is not an alternating series. Look carefully to see if you can tell why not.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)\, n}{n^2+1}$$

10.5 Alternating Series Test

Calculus

Practice

1. Explain why the Alternating Series Test does not apply to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$.

2. The Alternating Series Test can be used to show convergence of which of the following alternating series?

I. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ II. $\sum_{n=2}^{\infty} (-1)^{n+1}\left(\frac{n}{n^2+4}\right)$ III. $\sum_{n=1}^{\infty} (-1)^{n}\left(\frac{4n}{5n+3}\right)$

A. I only B. II only C. III only

D. I and II only E. I, II, and III

3. Which of the following series converge?

A. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1-2n}{n}\right)$

B. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{3n}\right)$

C. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^3}{2\sqrt{n}}\right)$

D. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2\sqrt{n}}{n^3}\right)$

Use the Alternating Series Test to show the series are convergent.

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^2}\right)$

5. $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{1}{3^n}\right)$

6. **Calculator active.** Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{(-1)^n+\sqrt{n}}$

I. The series is alternating.
II. $|a_{n+1}| \leq |a_n|$ for $n \geq 2$.
III. $\lim_{n\to\infty} a_n = 0$

A. I only B. I and II only C. I and III only D. I, II, and III

---

7. **Calculator active.** Which of the following statements about the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n = \frac{2+\cos n}{n^2}$ is true?

A. The series converges by the Alternating Series Test

B. The Alternating Series Test cannot be used because the series is not alternating.

C. The Alternating Series Test cannot be used because $\lim_{n\to\infty} a_n \neq 0$.

D. The Alternating Series Test cannot be used because the terms of a_n are not decreasing.

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8. The Alternating Series Test can be used to show convergence for which of the following series?

A. $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots$, where $a_n=\frac{(-1)^{n+1}(n+1)}{n}$.

B. $\frac{2}{1}-\frac{1}{1}+\frac{2}{2}-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}+\frac{2}{4}-\frac{1}{4}+\cdots$

C. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots$, where $a_n=(-1)^{n+1}\frac{1}{n^2}$

D. $\frac{3}{2}-\frac{2}{2}+\frac{3}{3}-\frac{2}{3}+\frac{3}{4}-\frac{2}{4}+\cdots$

---

9. For which of the following series can the Alternating Series Test not be used?

A. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}$

B. $\sum_{n=2}^{\infty}\frac{(-1)^n\ln(n^3)}{n}$

C. $\sum_{n=4}^{\infty}\frac{(-1)^n n}{n-3}$

D. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

---

10. Which of the following statements about the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}$ is true?

A. The series diverges by comparison to $\frac{1}{n}$.

B. The series converges by comparison to $\frac{1}{n}$.

C. The series diverges by the Alternating Series Test.

D. The series converges by the Alternating Series Test.

11. Which of the following statements are true about the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)!}{(n)!}$?
   I. The series is alternating.
   II. $|a_{n+1}| \leq |a_n|$ for $n \geq 1$.
   III. $\lim_{n\to\infty} a_n = 0$

A. I only B. I and II only C. I and III only D. I, II, and III

10.5 Alternating Series Test

Test Prep

12. The Alternating Series Test can be used to show convergence for which of the following series?

   I. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n^2}\right)$

   II. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n^2}$

   III. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{4}-1} + \cdots\right)$

A. I only B. I and II only C. II and III only D. I, II, and III

---

13. If $\sum_{n=1}^{\infty} \frac{(-1)^n}{a_n}$ converges, which of the following must be true?

A. $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \geq a_n > 0$ for $n \geq 1$.

B. $\lim_{n\to\infty} a_n = \infty$ and $a_{n+1} \leq a_n$ for $n \geq 1$.

C. $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \leq a_n$ for $n \geq 1$.

D. $\lim_{n\to\infty} a_n = \infty$ and $a_{n+1} \geq a_n > 0$ for $n \geq 1$.

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14. For what value of $k > 0$ will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{6}{k}\right)^n$ diverge?

A. 3

B. 4

C. 5

D. 7

Write your questions and thoughts here!

Find the values of x that make the series converge absolutely.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n(x+4)^n}{6^n}$

5. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

10.5 Absolute or Conditional Convergence

Calculus

Practice

1. Which of the following series are conditionally convergent?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

A. I only B. I and II only C. I and III only D. II and III only

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Determine whether the series converges absolutely, converges conditionally, or diverges.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2+8)}{\pi^n}$

3. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^2}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{5}{2}}}$

6. For which values x is the series $\sum_{n=1}^{\infty} \frac{nx^n}{4^n(n^2+1)}$ conditionally convergent?

A. $x = 4$

B. $x = -4$

C. $x > 4$

D. $-4 < x < 4$

7. Which of the following statements is true about the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/3}}$.

A. The series converges conditionally.

B. The series converges absolutely.

C. The series converges but neither conditionally nor absolutely.

D. The series diverges.

---

8. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^{3/2}}$ is true?

A. The series converges conditionally.

B. The series converges absolutely.

C. The series converges but neither conditionally nor absolutely.

D. The series diverges.

---

9. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ is true?

I. Converges Absolutely

II. Diverges

III. Converges Conditionally

A. I only

B. II only

C. III only

D. I and III only

---

10. For what values of x is the series $\sum_{n=1}^{\infty} \frac{n(x+5)^n}{7^n}$ absolutely convergent?

A. $x = -12$

B. $x = 2$

C. $x > 2$

D. $-12 < x < 2$

No test prep for this lesson.

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10.5 AP Practice Problems (p.765) – Alternating Series Test & Absolute Convergence

1. Which of the following series converge?

I. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$

II. $\sum_{k=1}^{\infty} (-1)^k \left(\frac{5}{3}\right)^k$

III. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

2. The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{10^k}$ converges. What is the maximum error incurred by using the first three nonzero terms to approximate the sum of the series?

(A) −0.083 (B) 0.003 (C) 0.0004 (D) 0.0826

3. What is the fewest number of terms of the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ that must be added to approximate the sum so that the error is less than or equal to 0.001?

(A) 7 (B) 9 (C) 10 (D) 11

4. Which of the following series converge conditionally, but not absolutely?

I. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{k}$

II. $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k}\right)^{4/3}$

III. $\sum_{k=0}^{\infty} (-1)^{k} \left(\frac{3}{4}\right)^{k}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

5. (a) Write out the first five terms of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$.

(b) Show the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ converges.

AGI 58 (c) How many terms of the series are necessary to approximate the sum S with an error less than or equal to 0.0001?

6. Determine whether the series $\sum_{k=1}^{\infty} \frac{\cos(2k)}{4^k}$ converges absolutely, converges conditionally, or diverges. Show your work.

BC Calculus – 10.6 Notes – Ratio Test and Root Test

Recall:

$\frac{(n+1)!}{n!} =$

$\frac{3^{n+1}}{3^n} =$

Ratio Test for Convergence

If $\sum_{n=1}^{\infty} a_n$ has positive terms and…

- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then the series
- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then the series
- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then

Let's look at two series we already know.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Using the Ratio Test to find convergence or divergence.

3. $\sum_{n=1}^{\infty} \frac{n^2 \cdot 3^{n+1}}{5^n}$

4. $\sum_{n=1}^{\infty} \frac{4^n}{n!}$

5. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

The final test to determine convergence or divergence is the root test. The root test is especially well suited to solve series involving n^{th} powers.

Root Test

Let $\sum a_n$ be a series.

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2) $\sum a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3) The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Example: Using the Root Test

6) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

7) $\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^n$

8) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

9) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

10) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

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Write your questions and thoughts here!

5. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

10.6 Ratio Test

Calculus

Practice

Determine whether the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{(n+1)3^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

3. What are values of $x > 0$ for which the series $\sum_{n=1}^{\infty} \frac{n6^n}{x^n}$ converges?

4. What are all positive values of p for which the series $\sum_{n=1}^{\infty} \frac{n^p}{7^n}$ will converge?

A. $p > 0$B. $0 < p < 7$

C. $p > 1$D. There are no positive values where the series will converge.

5. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{7^n}{n!}$ II. $\sum_{n=1}^{\infty} \frac{n!}{n^{20}}$ III. $\sum_{n=1}^{\infty} \frac{\pi^{-2n}}{n}$

A. I only B. I and II only C. III only D. I and III only E. I, II, and III

---

6. If the Ratio Test is applied to the series $\sum_{n=1}^{\infty} \frac{n\pi^n}{15^n}$, which of the following inequalities results, implying that the series converges?

A. $\lim_{n\to\infty} \frac{n\pi^n}{15^n} < 1$ B. $\lim_{n\to\infty} \frac{15^n}{n\pi^n} < 1$ C. $\lim_{n\to\infty} \frac{(n+1)\pi^{n+1}}{15^{n+1}} < 1$ D. $\lim_{n\to\infty} \frac{(n+1)\pi}{15n} < 1$

---

7. If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 6$, which of the following series converges?

A. $\sum_{n=1}^{\infty} a_n$ B. $\sum_{n=1}^{\infty} \frac{a_n}{n^7}$ C. $\sum_{n=1}^{\infty} \frac{a_n}{7^n}$ D. $\sum_{n=1}^{\infty} \frac{(a_n)^2}{7^n}$

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8. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{3^n}$. If the Ratio Test is applied to the series, which of the following inequalities results, implying the series diverges?

A. $\lim_{n\to\infty} \frac{n!}{3^n} < 1$ B. $\lim_{n\to\infty} \frac{n!}{3^n} > 1$ C. $\lim_{n\to\infty} \frac{n+1}{3} < 1$ D. $\lim_{n\to\infty} \frac{n+1}{3} > 1$

---

9. For which of the series is the Ratio Test inconclusive?

I. $\sum_{n=1}^{\infty} \frac{1}{3n}$ II. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ III. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

A. I only B. II only C. I and III only D. I and II only E. I, II, and III

---

10. Apply any appropriate test to determine which of the following series diverges.

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$ II. $\sum_{n=1}^{\infty} \frac{n!}{9^n}$ III. $\sum_{n=1}^{\infty} \frac{n+1}{4n+1}$

A. I only B. II only C. III only D. I and II only E. I, II, and III

Match the test for convergence of an infinite series with the conditions of convergence.

Convergence Test	Condition of convergence
11. _____ nth-Term Test	A. $p > 1$
12. _____ Geometric Series	B. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$
13. _____ p-series	C. $0 < a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges
14. _____ Alternating Series Test	D. $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges
15. _____ Integral Test	E. $\lvert r \rvert < 1$
16. _____ Ratio Test	F. Inconclusive for convergence
17. _____ Comparison Test	G. $\lvert a_{n+1} \rvert \le \lvert a_n \rvert$ and $\lim_{n\to\infty} a_n = 0$
18. _____ Limit Comparison Test	H. $\int_1^{\infty} f(x)\,dx$ converges.

10.6 Ratio Test

Test Prep

19. If the Ratio Test is applied to the series $\sum_{n=1}^{\infty} \frac{7^n}{(n+1)!}$, which of the following limits results, implying that the series converges?

A. $\lim_{n\to\infty} \frac{7^n}{(n+1)!}$ B. $\lim_{n\to\infty} \frac{7}{n+2}$ C. $\lim_{n\to\infty} \frac{(n+1)!}{7^n}$ D. $\lim_{n\to\infty} \frac{n+2}{7}$

20. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

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10.6 AP Practice Problems (p. 772) – Ratio Test & Root Test

1. Which of the following series converge?

   I. $\sum_{k=1}^{\infty} \frac{k^2}{(3k+1)!}$

   II. $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

   III. $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

   (A) II only (B) III only
   (C) I and III only (D) II and III only

2. For which of the following series does the Ratio Test provide no information?

   I. $\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k}}{k+1}$ II. $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ III. $\sum_{k=1}^{\infty} \frac{k^2+3k}{k+1}$

   (A) I only (B) II only
   (C) I and II only (D) I, II, and III

3. Which of the following series diverge?

I. $\sum_{k=1}^{\infty} \frac{k!}{100^k}$

II. $\sum_{k=1}^{\infty} \frac{20^k}{2^{k^2}}$

III. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

4. Determine whether $\sum_{k=1}^{\infty} \frac{e^k}{k^k}$ converges or diverges. Show your work.

5. Show that for any p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p > 0$, the Ratio Test provides no information about whether the series converges or diverges.

BC Calculus – 10.7 Notes – Summary of Convergence Tests

TABLE 5 Tests for Convergence and Divergence of Series

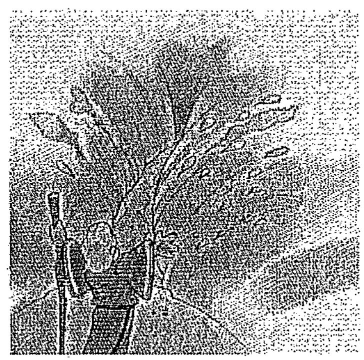
Test Name	Description	Comment
nth Term Test for Divergence for all series (p. 735)	$\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} a_n \neq 0$.	No information is obtained about convergence if $\lim_{n\to\infty} a_n = 0$.
Integral Test for series of positive terms (p. 738)	$\sum_{k=1}^{\infty} a_k$ converges (diverges) if $\int_1^{\infty} f(x)\ dx$ converges (diverges), where f is continuous, positive, and nonincreasing for $x \geq 1$; and $f(k) = a_k$ for all k.	Good to use if f is easy to integrate.
Comparison Test for Convergence for series of positive terms (p. 747)	$\sum_{k=1}^{\infty} a_k$ converges if $0 < a_k \leq b_k$ and the series $\sum_{k=1}^{\infty} b_k$ converges.	$\sum_{k=1}^{\infty} b_k$ must have positive terms and be convergent.
Comparison Test for Divergence for series of positive terms (p. 747)	$\sum_{k=1}^{\infty} a_k$ diverges if $a_k \geq c_k > 0$ and the series $\sum_{k=1}^{\infty} c_k$ diverges.	$\sum_{k=1}^{\infty} c_k$ must have positive terms and be divergent.
Limit Comparison Test for series of positive terms (p. 748)	$\sum_{k=1}^{\infty} a_k$ converges (diverges) if $\sum_{k=1}^{\infty} b_k$ converges (diverges), and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, a positive real number.	$\sum_{k=1}^{\infty} b_k$ must have positive terms, whose convergence (divergence) can be determined.
Alternating Series Test (p. 755)	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k,\ a_k > 0,$ converges if • $\lim_{n\to\infty} a_n = 0$ and • the a_k are nonincreasing.	The error made by using the nth partial sum to approximate the sum S of the series is less than or equal to $\lvert a_{n+1} \rvert$.
Absolute Convergence Test (p. 759)	If $\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.	The converse is not true. That is, if $\sum_{k=1}^{\infty} \lvert a_k \rvert$ diverges, $\sum_{k=1}^{\infty} a_k$ may or may not converge.
Ratio Test for series with nonzero terms (p. 765)	$\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$. $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ or if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \infty$.	Good to use if a_n includes factorials or powers. It provides no information if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$ or if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert \neq \infty$ does not exist.
Root Test for series with nonzero terms (p. 768)	$\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$. $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$ or if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = \infty$.	Good to use if a_n involves nth powers. It provides no inform. $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$.

TABLE 6 Important Series

Series Name	Series Description	Comments
Geometric series (pp. 725–726)	$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots, \quad a \neq 0$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
Harmonic series (p. 729)	$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$	Diverges.
p-series (p. 740)	$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$	Converges if $p > 1$; diverges if $0 < p \leq 1$.
k-to-the-k series (p. 747)	$\sum_{k=1}^{\infty} \frac{1}{k^k} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$	Converges.
Alternating harmonic series (p. 756)	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$	Converges.

So, how can you remember all these tests (besides using your Jedi powers)? Try this Moses phrase:

PARTING C



P p-series: Is the series in the form $\frac{1}{n^p}$?

A Alternating series: Does the series alternate? If it does, are the terms getting smaller, and is the nth term 0?

R Ratio Test: Does the series contain things that grow very large as n increases (exponentials or factorials)?

T Telescoping series: Will all but a couple of the terms in the series cancel out?

I Integral Test: Can you easily integrate the expression that defines the series (are Dogs Cussing in Prison?)

N nth Term divergence Test: Is the nth term something other than zero?

G Geometric series: Is the series of the form $\sum_{n=0}^{\infty} ar^n$?

C Comparison Tests: Is the series ***almost*** another kind of series (e.g. p-series or geometric)? Which would be better to use: the Direct or Limit Comparison Test?

Summary of Tests for Infinite Series Convergence

Given a series

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum_{n=0}^{\infty} a_n$$

The following is a summary of the tests that we have learned to tell if the series converges or diverges. They are listed in the order that you should apply them, unless you spot it immediately, i.e. use the first one in the list that applies to the series you are trying to test, and if that doesn't work, try again. Off you go, young Jedis. Use the Force. Remember, it is always with you, and it is mass times acceleration!

_n_th-term test: (Test for Divergence only)

If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent. If $\lim_{n\to\infty} a_n = 0$, then the series may converge or diverge, so you need to use a different test.

Geometric Series Test:

If the series has the form $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^n$, then the series converges if $|r| < 1$ and diverges otherwise. If the series converges, then it converges to $\frac{a_1}{1-r}$.

Integral Test:

In ***P***rison, ***D***ogs ***C***urse: If $a_n = f(n)$ is ***P***ositive, ***D***ecreasing, ***C***ontinuous function, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(n)dn$ either both converge or both diverge.

- This test is best used when you can easily integrate a_n.
- ***Careful***: If the Integral converges to a number, this is NOT the sum of the series. The series will be smaller than this number. We only know this it also converges, to what is anyone's guess.
- The maximum error, R_n, for the sum using S_n will be $0 \leq R_n \leq \int_n^{\infty} f(x)dx$

***p*-series test:**

If the series has the form $\sum \frac{1}{n^p}$, then the series converges if $p > 1$ and diverges otherwise. When $p = 1$, the series is the divergent Harmonic series.

Alternating Series Test:

If the series has the form $\sum(-1)^n a_n$, then the series converges if $0 < a_{n+1} \le a_n$ (decreasing terms) for all n, for some n, and $\lim_{n\to\infty} b_n = 0$. If either of these conditions fails, the test fails, and you need use a different test.

- if the series converges, the sum, S, lies between $S_n - a_{n+1}$ and $S_n + a_{n+1}$
- if $\left|\sum a_n\right|$ converges then $\sum a_n$ is *Absolutely Convergent*
- if $\left|\sum a_n\right|$ diverges but $\sum a_n$ converges, then $\sum a_n$ is *Conditionally Convergent*
- if $\left|\sum a_n\right|$ converges, then $\sum a_n$ converges.

Direct Comparison Test:

If the series looks like another series $\sum b_n$, then:

- If $a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges also.
- If $a_n \ge b_n$ and $\sum b_n$ diverges then $\sum a_n$ diverges also.

You need to know if $\sum b_n$ converges or diverges, so you usually use a geometric series, p-series, or integrable series for the comparison. You must verify that for sufficiently large values of n, the rule of sequence of one is greater than or equal to the other term for term. Use this test when the rule of sequence if VERY SIMILAR to a known series.

Ex) compare $\frac{n}{2^n}$ to $\frac{1}{2^n}$, $\frac{1}{n^3+1}$ to $\frac{1}{n^3}$, $\frac{n^2}{\left(n^2+3\right)^2}$ to $\frac{n}{\left(n^2+3\right)^2}$

Limit Comparison Test:

(may be used instead of Direct Comparison Test most of the time)

If a_n, $b_n > 0$ and $\lim_{x\to\infty}\left|\frac{a_n}{b_n}\right|$ <u>**or**</u> $\lim_{x\to\infty}\left|\frac{b_n}{a_n}\right|$ equal any finite number, then either both $\sum a_n$ and $\sum b_n$ converge or diverge.

Use this test when you cannot compare term by term because the rule of sequence is "too UGLY" but you can still find a known series to compare with it.

Ex) compare: $\frac{3n^2+2n-1}{4n^5-6x+7}$ to $\frac{1}{n^3}$ (you can disregard the leading coefficient and all non-leading terms, looking only at the condensed degree of the leading terms: $\frac{n^2}{n^5} = \frac{1}{n^3}$.

Ratio Test:

If $a_n > 0$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = N$ (where N is a real number), then

1. $\sum a_n$ converges absolutely (and hence converges) if $N < 1$
2. $\sum a_n$ diverges if $N > 1$ or $N = \infty$
3. The test is inconclusive if $N = 1$ (use another test)

Use this test for series whose terms converge rapidly, for instance those involving exponentials and/or factorials!!!!!!!

Root Test:

If $\sum a_n$ is a series with non-zero terms and $\lim_{n\to\infty}\sqrt[n]{|a_n|} = N$ (where N is a real number), then

1. $\sum a_n$ converges absolutely (and hence converges) if $N < 1$
2. $\sum a_n$ diverges if $N > 1$ or $N = \infty$
3. The test is inconclusive if $N = 1$ (use another test)

Use this test for series involving nth powers. Ex) $\sum \frac{e^{2n}}{n^n}$

Remember, if you are asked to find the ACTUAL sum of an infinite series, it must either be a Geometric series $\left(S = \frac{a_1}{1-r}\right)$ or a Telescoping Series (requires expanding and canceling terms). The telescoping series can be quite overt, such as $\sum\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$ or in "disguise" as $\sum \frac{2}{4n^2-1}$, in which case partial fraction decomposition must be used. Also note that it is possible to tell that this last series converges by Comparison tests, but the actual sum can only be given by expanding!

The only other tests that allows us to *approximate* the infinite sum are the Integral test and the Alternate Series Test. We can find the nth partial Sum S_n for any series.

$\sum_{n=1}^{\infty} a_n$ Convergence / Divergence Flow Chart for 10.1 through 10.9

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Nth Term Test for Divergence

Does $\lim_{n\to\infty} a_n = 0$?
- No → $\sum_{n=1}^{\infty} a_n$ Diverges
- Yes ↓

p-series

Does $a_n = \frac{1}{n^p}, n \geq 1$?
- Yes → Is $p > 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges
- No ↓

Geometric Series

Does $a_n = ar^{n-1}, n \geq 1$?
- Yes → Is $|r| < 1$?
  - Yes → $\sum_{n=k}^{\infty} a_n = \frac{ar^k}{1-r}$
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges
- No ↓

Alternating Series

Does $a_n = (-1)^n b_n$, or $a_n = (-1)^{n+1} b_n$, $b_n \geq 0$?
- Yes → Is $\lim_{n\to\infty} b_n = 0$ and $\frac{b_{n+1}}{b_n} < 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
- No ↓

Try one of the following:

Comparison Test

Pick b_n to compare to a_n. Does $\sum b_n$ converge?
- Yes → Is $0 \leq a_n \leq b_n$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → No conclusion can be made.
- No → Is $0 \leq b_n \leq a_n$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Diverges

Limit Comparison Test

Pick b_n. Does $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ and finite, and $a_n, b_n > 0$?
- Yes → Does $\sum_{n=1}^{\infty} b_n$ converge?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

Integral Test

Is $a_n = f(n)$ continuous, positive and decreasing on $[a, \infty)$?
- Yes → Does $\int_a^{\infty} f(n)\, dn$ converge?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

Ratio Test

Is $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \neq 1$?
- Yes → Is $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Abs. Conv.
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

ROOT TEST

Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} \neq 1$?
- YES → Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$?
  - YES → $\sum a_n$ Abs. Conv.
  - NO → $\sum a_n$ Diverges