Key

Ch. 10 Unit Review AP Practice Problems (p.813) – Infinite Series

1. Suppose $f(2) = 3$; $f'(2) = 0$; $f''(2) = 5$; $f'''(2) = -4$; and $f^{(4)}(2) = -2$. Then the Taylor polynomial $P_4(x)$ of degree 4 of f at 2 is

(A) $P_4(x) = 3(x-2) + \frac{5}{2}(x-2)^2 - \frac{4}{6}(x-2)^3 - \frac{2}{24}(x-2)^4$

(B) $P_4(x) = 3 + \frac{5}{2}(x-2)^2 - \frac{2}{3}(x-2)^3 - \frac{1}{12}(x-2)^4$

(C) $P_4(x) = 3 + (x-2) + \frac{5}{2}(x-2)^2 - \frac{2}{3}(x-2)^3 + \frac{1}{12}(x-2)^4$

(D) $P_4(x) = 3 + \frac{5}{2!}(x-2)^2 - \frac{4}{3!}(x-2)^3 + \frac{2}{4!}(x-2)^4$

Taylor polynomial:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

$$P_4(2) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^4(2)}{4!}(x-2)^4$$

$$= 3 + 0(x-2) + \frac{5}{2}(x-2)^2 + \frac{-4}{6}(x-2)^3 + \frac{-2}{24}(x-2)^4$$

2. If $0 < a_k \le b_k$ for all k and $\sum_{k=1}^{\infty} a_k$ diverges, then which statement must be false?

(A) $\lim_{n\to\infty} a_n = 0$ (B) $\lim_{n\to\infty} b_n = 0$

(C) $\sum_{k=1}^{\infty} b_k = 1$ (D) $\sum_{k=1}^{\infty} (-1)^k a_k$ diverges

*If the smaller series diverge, then the larger series diverge as well.

3. Which of the following series converge?

I. $\sum_{k=1}^{\infty} k^{-3/2}$ II. $\sum_{k=1}^{\infty} k^{-1}$ III. $\sum_{k=1}^{\infty} 2^{1-k}$

(A) I only (B) III only

(C) I and III only (D) I, II, and III

iii) $\sum \frac{2^1}{2^k} \to \sum 2\left(\frac{1}{2}\right)^k$

$r = \frac{1}{2} < 1$ converges by GST

i) $\sum \frac{1}{k^{3/2}} \to p = 3/2 > 1$ converges

ii) $\sum \frac{1}{k}$ harmonic p-series diverge

4. Determine whether the infinite series $\sum_{k=1}^{\infty}\left(-\frac{5}{6}\right)^{k-1}$ converges or diverges. If the series converges, find its sum S.

(A) converges; $S = -\frac{5}{11}$ $\boxed{\text{(B)}}$ converges; $S = \frac{6}{11}$

(C) converges; $S = 6$ (D) The series diverges.

$\sum \left(\frac{-5}{6}\right)^k \cdot \left(\frac{-5}{6}\right)^{-1}$

$\sum \frac{-6}{5}\left(\frac{-5}{6}\right)^k$

By GST, series converge since $|r| = |-5/6| < 1$

$S = \frac{a_1}{1-r} \rightarrow \frac{1}{1-(-5/6)} \rightarrow \frac{1}{1+5/6} \rightarrow \frac{1}{11/6}$

$\boxed{S = \frac{6}{11}}$

5. The interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$ is

(A) $-1 < x < 1$ (B) $-1 < x \leq 1$

(C) $1 \leq x < 3$ $\boxed{\text{(D)}}$ $1 \leq x \leq 3$

$\rightarrow \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

*Ratio Test

$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-2)^n}(x-2)^1\cdot n^2}{\cancel{(x-2)^n}\cdot(n+1)^2}\right| < 1$

$|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

Test $x=1$:

$\sum \frac{(1-2)^n}{n^2} \rightarrow \frac{(-1)^n}{n^2}$ converges $p=2$

Test $x=3$:

$\sum \frac{(3-2)^n}{n^2} = \frac{1^n}{n^2}$

$\boxed{\text{IOC } 1 \leq x \leq 3}$

6. For which integer n do all three infinite series converge?

$$\sum_{k=1}^{\infty} k^{-n/3} \qquad \sum_{k=1}^{\infty} \frac{(-1)^{nk}}{k} \qquad \sum_{k=1}^{\infty}\left(\frac{n}{6}\right)^k$$

(A) 2 (B) 3 (C) 4 $\boxed{\text{(D)}}$ 5

i) $\sum \left(\frac{1}{k}\right)^{n/3}$

$\sum \left(\frac{1}{k^{n/3}}\right)$

p-series, $n > 3$ in order to converge

ii) $\sum \frac{(-1)^{nk}}{k}$

converges if alternating series. n must be an odd number

iii) $\sum \left(\frac{n}{6}\right)^k$ → geometric series

$\frac{n}{6} < 1$, $\underline{n < 6}$

$\boxed{\text{To satisfy all 3 conditions, } n = 5}$

7. If a function f is continuous, positive, and decreasing on the interval $[1, \infty)$ and if $a_k = f(k)$ for all positive integers k, then the infinite series

(A) $\sum_{k=1}^{\infty} a_k$ converges.

(B) $\sum_{k=1}^{\infty} a_k = \int_1^{\infty} f(x)dx$ if $\int_1^{\infty} f(x)dx$ converges.

(C) $\sum_{k=1}^{\infty} a_k = f(x)$

(D) $\sum_{k=1}^{\infty} a_k$ converges if $\int_1^{\infty} f(x)dx$ converges. $\leftarrow$ condition for Integral Test.

$e^1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n!}$

$\frac{1}{k!} = \frac{1}{k} + \frac{1}{(k-1)} + \frac{1}{(k-2)} \cdots$

8. $\sum_{k=1}^{\infty} \frac{3^{k+1} + 4k!}{3^k \cdot k!} =$

(A) $e + 2$ (B) $3e - 1$ (C) $3e + \frac{8}{3}$ (D) $3e + 2$

* split into 2 series

$\sum \frac{3^{k+1}}{3^k \cdot k!} + \frac{4\cancel{k!}}{3^k \cdot \cancel{k!}} \rightarrow \frac{\cancel{3^k} \cdot 3}{\cancel{3^k} \cdot k!} + \frac{4}{3^k}$

$\sum \frac{3}{k!} + \frac{4}{3^k}$

$3\sum \frac{1}{k!} + 4\sum \left(\frac{1}{3}\right)^k$

$3\left(\left(\sum_{k=0}^{\infty} \frac{1}{k!}\right) - 1\right) + 4\sum_{k=1}\left(\frac{1}{3}\right)^k$

$3(e-1) + 4\left(\frac{1/3}{1-1/3}\right)$

$3e - 3 + \left(\frac{4}{3}\right)\left(\frac{3}{2}\right)$

$3e - 3 + 2 = \boxed{3e-1}$

9. Find bounds for the sum of $\sum_{k=1}^{\infty} \frac{3}{(2k)^4}$. $\rightarrow \frac{3}{2^4 k^4} \rightarrow \frac{3}{16}\left(\frac{1}{k^4}\right)$

(A) $\frac{1}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{1}{4}$ (B) $\frac{1}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{17}{16}$

(C) $\frac{3}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{4}{3}$ (D) $\frac{1}{3} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{4}{3}$

* Sum of convergent p-series:

$\frac{1}{p-1} < \sum \frac{1}{x^p} < 1 + \frac{1}{p-1}$

$\frac{3}{16}\left[\frac{1}{4-1} < \sum \frac{1}{k^4} < 1 + \frac{1}{4-1}\right]$

$\frac{3}{16}\left(\frac{1}{3}\right) < \frac{3}{16}\sum \frac{1}{k^4} < \frac{3}{16}\left(1 + \frac{1}{3}\right)$

$\boxed{\frac{1}{16} < \frac{3}{16}\sum \frac{1}{k^4} < \frac{4}{16} \text{ or } 1/4}$

10. Which of the series diverge?

geometric series $r = -1/2$ (convergent)

I. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$

II. $\frac{2}{1^{3/2}} + \frac{4}{2^{3/2}} + \frac{6}{3^{3/2}} + \frac{8}{4^{3/2}} + \cdots$

III. $6 - 4 - \frac{8}{3} - \frac{16}{9} - \frac{32}{27} - \cdots$

$r = 4/6$ or $\frac{8/3}{-4} \rightarrow \frac{8}{3} \cdot \frac{1}{4} = -\frac{2}{3}$ (converge)

(A) I only $\boxed{\text{(B)}}$ II only

(C) I and II only (D) I and III only

ii) $\sum_{k=1}^{\infty} \frac{2k}{k^{3/2}} \rightarrow \sum \frac{2}{k^{1/2}}$ $p = 1/2 < 1$ $\boxed{\text{diverges}}$

11. The interval of convergence of the power series $\sum_{k=0}^{\infty} \left(\frac{x-1}{4}\right)^k$ is

$\rightarrow \sum_{n=0}^{\infty} \left(\frac{x-1}{4}\right)^n$

(A) $-1 \le x \le 1$ (B) $-4 < x < 4$

(C) $-4 < x \le 4$ $\boxed{\text{(D)}}$ $-3 < x < 5$

*Ratio Test

$$\lim_{n\to\infty} \left| \left(\frac{x-1}{4}\right)^{n+1} \cdot \left(\frac{4}{x-1}\right)^n \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^n}(x-1)\cdot \cancel{4^n}}{\cancel{4^n}\cdot 4^1 \cancel{(x-1)^n}} \right| < 1$$

$\left|\frac{x-1}{4}\right| < 1$

$|x-1| < 4$

$-4 < x-1 < 4$

$-3 < x < 5$

test $x = -3$: $\sum \left(\frac{-4}{4}\right)^k$ diverge

test $x = 3$: $\sum \left(\frac{4}{4}\right)^k$ diverge

$\boxed{\text{IOC: } -3 < x < 5}$

12. Which series is the Maclaurin expansion of $f(x) = x^2 \sin x$?

(A) $x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \frac{x^8}{6!} + \cdots$

(B) $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

$\boxed{\text{(C)}}$ $x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots$

(D) $\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$

$f(x) = x^2 \sin x = x^2\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right]$

$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \frac{x^{11}}{9!}$

13. Determine whether the series $\sum_{k=1}^{\infty} \frac{4^{k+2}}{5^k}$ converges. If it converges, its sum equals

(A) converges; 5 (B) converges; 64
(C) converges; 80 (D) The series diverges.

$\sum \frac{4^k \cdot 4^2}{5^k} \rightarrow 16 \sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k$

$S = \frac{a_1}{1-r}$ | $a_1 = \frac{4^3}{5}$, $r = \frac{4}{5}$

$S = \frac{64/5}{1-4/5} = \frac{64/5}{1/5}$

$S = \frac{64}{5} \cdot \frac{5}{1} = \boxed{64}$

14. Determine whether the series $\sum_{k=1}^{\infty} \frac{6}{(k+1)(k+2)}$ converges. If it converges, its sum equals

(A) converges; 1 (B) converges; 3
(C) converges; 6 (D) The series diverges.

*split up using partial fraction

$\frac{6}{(k+1)(k+2)} = \frac{A}{k+1} + \frac{B}{k+2}$ $A=6$ $B=-6$

$(k=-1)$ $(k=-2)$

$\sum \frac{6}{k+1} + \frac{-6}{k+2}$ ← converging telescoping series

$\left(\frac{6}{2} - \frac{6}{3}\right) + \left(\frac{6}{3} - \frac{6}{4}\right) + \ldots = \frac{6}{2} = \boxed{3}$

15. A Maclaurin polynomial for $f(x) = e^x$ is used to approximate $e^{1/3}$. What is the degree of the Maclaurin polynomial needed to ensure that the Lagrange error bound is less than 0.00001?

(A) 3 (B) 5 (C) 7 (D) 9

$R_n(x) \leq \left| \frac{\text{Max}\left[f^{n+1}(z)\right](x-c)^{n+1}}{(n+1)!} \right|$

$\rightarrow \frac{e^{1/3}(1/3 - 0)^{n+1}}{(n+1)!} < 0.00001$

$\frac{(1/3)^{n+1}}{(n+1)!} < 0.000007165$

when $n=3$, ≈ 0.0005
$n=4$, 0.0000343
$n=5$, 0.00000572

$\boxed{\text{degree of polynomial is 5}}$

16. A fourth degree Taylor polynomial for $f(x) = \sin x$ at $\frac{\pi}{6}$ is used to approximate $\sin 35°$. The Lagrange error bound guarantees the error in using the approximation is less than

(A) $\frac{1}{4!}\left(\frac{\pi}{36}\right)^4$ $\quad$ $\boxed{\text{(B)}}$ $\frac{1}{5!}\left(\frac{\pi}{36}\right)^5$

(C) $\frac{1}{5!}\left(\frac{\pi}{6}\right)^5$ $\quad$ (D) $\frac{1}{7!}\left(\frac{\pi}{6}\right)^7$

$35° = \frac{7\pi}{36}$

$\frac{7\pi}{36} - \frac{\pi}{6} = \frac{\pi}{36}$

$R_n(x) = \left| \frac{\text{Max } f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} \right|$

$R_4(x) = \left| \frac{1}{5!} \cdot \left(\frac{\pi}{36}\right)^5 \right|$

17. Determine whether the infinite series $\sum_{k=1}^{\infty} \frac{(\ln k)^2}{k}$ converges or diverges. Be sure to show all your work.

*Integral Test: $f(x) = \frac{(\ln x)^2}{x}$ → $f(x)$ is continuous, positive, decreasing $(1, \infty)$

$\int_1^\infty \frac{(\ln x)^2}{x}\,dx$

$u = \ln x$ | $dx = x\,du$

$du/dx = \frac{1}{x}$

$\int \frac{u^2}{x} \cdot x\,du \rightarrow \frac{u^3}{3} \rightarrow \frac{1}{3}(\ln x)^3 \Big]_1^b \rightarrow \lim_{b\to\infty} \frac{1}{3}(\ln b)^3 - \frac{1}{3}(\ln 1)^3 = \infty$

Since the integral diverges, the series also diverge.

18. Determine whether the infinite series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$ is absolutely convergent, conditionally convergent, or divergent. Show your work.

*limit comparison test: use $\frac{1}{k^{1/2}}$ (diverging p-series)

$\lim_{k\to\infty} \frac{k^{1/2}}{k+1} \cdot \frac{k^{1/2}}{1} = 1$ (series also diverges) → conditionally convergent

since series converge by AST $\lim_{k\to\infty} \frac{\sqrt{k}}{k+1} = 0$

19. (a) Write the first five nonzero terms of the Maclaurin expansion of $f(x) = \tan^{-1} x$.

(b) Use properties of power series to obtain the Maclaurin series for $g(x) = \int_0^x \tan^{-1} t \, dt$.

(c) Using the fact that the radius of convergence R of the Maclaurin series representation for $f(x) = \tan^{-1} x$ is 1, find the interval of convergence of the series representation of g.

(d) Use the result from (b) to approximate $\int_0^{1/4} \tan^{-1} x \, dx$ so the error is less than 0.001.

a) $f(x) = \tan^{-1}(x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\quad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\quad f''(0) = 0$

$f'''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$ $\quad f'''(0) = -2$

$f^4(x) = \frac{24x - 24x^3}{(1+x^2)^4}$ $\quad f^4(0) = 0$

$f^5(x) = \frac{120x^4 - 240x^2 + 24}{(1+x^2)^5}$ $\quad f^5(0) = 24$

$f^6(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(1+x^2)^6}$ $\quad f^6(0) = 0$

Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n \Rightarrow 0 + 1x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{24}{5!}x^5 + \frac{0}{6!}x^6$$

First 5 terms: $\boxed{P_9(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9}$

General Term

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

b) $\int_0^x \tan^{-1}(t)\,dt \approx \left.\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{5\cdot 6} - \frac{t^8}{7\cdot 8} + \ldots\right]_0^x = \boxed{\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} \ldots + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k)(2k-1)}}$

c) test $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} \rightarrow$ IOC $\boxed{(-1, 1)}$

d) $|E_n| \leq a_{n+1} = \frac{\left(\frac{1}{4}\right)^{2(k+1)}}{(2(k+1))(2(k+1)-1)}$

1st unused term $\quad x = 1/4 \rightarrow$

$n = 1 \rightarrow a_2 = 0.000977$

$n = 2 \rightarrow a_3 = 0.000136$

$P_2(1/4) = \frac{(1/4)^2}{2} - \frac{(1/4)^3}{12} = \frac{95}{3072} \approx \boxed{0.031}$