and A has a local maximum at e^{-1}. Because $A''(x) < 0$ for all $0 < x < 1$, the local maximum is also an absolute maximum. Therefore, the area of the largest rectangle in the fourth quadrant that has three vertices on the coordinate axes and the fourth vertex on the graph of $y = \ln x$ is

$$A = -e^{-1}\ln e^{-1} = e^{-1} = \boxed{\frac{1}{e}}.$$

AP® Review Problems: Chapter 5

1. From the graph of $f(x)$ it appears that for $x < 0$, $f'(x) < 0$; for $0 < x < 1.4$, $f'(x) > 0$; and for $x > 1.4$, $f'(x) < 0$. $f'(x) = 0$ at $x = 0$ and $x \approx 1.4$.

   $\boxed{\text{The only graph of } f \text{ to satisfy these conditions is graph B}}$.

   CHOICE B

2. $f(x) = x^3 - 2x^2 + x$

   $f'(x) = 3x^2 - 4x + 1$

   Set $f'(x) = 3x^2 - 4x + 1 = 0$ to determine the endpoints of the intervals in which f is increasing or decreasing.

   $$f'(x) = 3x^2 - 4x + 1 = 0$$
   $$(3x - 1)(x - 1) = 0$$
   $$x = \frac{1}{3} \quad x = 1$$

Interval	Sign of $3x - 1$	Sign of $x - 1$	$f'(x) = 3x^2 - 4x + 1$	Conclusion
$\left(-\infty, \frac{1}{3}\right)$	$-$	$-$	$+$	Increasing
$\left(\frac{1}{3}, 1\right)$	$+$	$-$	$-$	Decreasing
$(1, \infty)$	$+$	$+$	$+$	Increasing

   $\boxed{f(x) = x^3 - 2x^2 + x \text{ is increasing on both } \left(-\infty, \frac{1}{3}\right] \text{ and } [1, \infty)}$.

   CHOICE C

3. $y(t) = te^{-t^2}$

   $y'(t) = e^{-t^2} + t(e^{-t^2}(-2t)) = e^{-t^2}(1 - 2t^2)$

   The object is at rest when $y'(t) = 0$

   $$y'(t) = e^{-t^2}(1 - 2t^2) = 0$$
   $$1 - 2t^2 = 0$$
   $$t = \pm\frac{\sqrt{2}}{2}$$

   For the domain of $t \geq 0$ the sole value for t is $\boxed{t = \frac{\sqrt{2}}{2}}$.

   CHOICE A

4. Consider each choice in turn.

   A. True. The table provides that $f''(2) = 2$ to the left of $f''(3) = 0$ indicating that f is concave up at $x = 2$ while $f''(4) = -2$ to the right of $f''(3) = 0$ indicating that f is concave down at $x = 4$.

   So $\boxed{f \text{ changes concavity on the interval } (2,4).}$

   B. False. f has a point of inflection at $x = 3$ if the concavity of f changes from the interval $(a, 3)$ to the interval $(3, b)$ for a and b in the domain. While the concavity of f changes on the interval $(2, 4)$ the table does not reveal where in $(2, 4)$ that the concavity changes. There may be other values of c in the interval $(2, 4)$ for which $f''(c) = 0$ resulting in a Point of Inflection in $(2, 4)$ other than at $x = 3$.

   C. False. f has a local maximum at a critical number. A critical number is determined when $f'(x) = 0$ or when $f'(x)$ does not exist. No information has been provided in the chart to determine where there is a critical number, if any.

   D. False. f is increasing on the interval $(0, 2)$ if $f'(x) > 0$ on $(0, 2)$. No information has been provided in the table to make that determination.

5. The Intermediate Value Theorem as applied to this problem provides that since f is continuous for all real numbers and since 0 is a y-coordinate between $f(-4) = 3$ and $f(1) = -8$ there is at least one number c in the domain such that $f(c) = 0$.

   CHOICE C

6. $f'(x) = x^{-2/3} - 4$

   Let $f'(x) = 0$ to determine the critical number(s), if any.

$$\begin{aligned} f'(x) = x^{-2/3} - 4 &= 0 \\ x^{-2/3} &= 4 \\ x &= \frac{1}{8} \end{aligned}$$

   Amongst possible methods to determine if f has a local minimum at $x = \frac{1}{8}$ is to apply the Second Derivative Test which is to determine the value of $f''(\frac{1}{8})$. If $f''(\frac{1}{8}) < 0$ then f is concave down at $x = \frac{1}{8}$ and there is a local maximum at $x = \frac{1}{8}$. If $f''(\frac{1}{8}) > 0$, as is the case here as shown below, then f is concave up at $x = \frac{1}{8}$ and there is a local minimum at $x = \frac{1}{8}$.

$$f'(x) = x^{-2/3} - 4$$

$$f''(x) = \frac{-2}{3}x^{-5/3} = \frac{-2}{3x^{5/3}}$$

$$f''\left(\frac{1}{8}\right) = \frac{-2}{3\left(\frac{1}{8}\right)^{5/3}} = \frac{-2}{\frac{3}{32}} = -\frac{64}{3} < 0$$

   The function f has a local minimum at $\boxed{x = \frac{1}{8}}$.

   CHOICE B

7. Consider each choice in turn.

   I. With the given information, f could be an even function greater than 2 and $f(c)$ is not necessarily equal to 0.

   II. Rolle's Theorem, a specific application of the Mean Value Theorem, provides that if f is a continuous function, as it is here being a polynomial function, and differentiable on the open interval (a, b) then if $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$. So,

   $\boxed{f'(c) = 0 \text{ is true.}}$

   III. $f''(c) = 0$ is not true as there can be a function f with the specified requirements without a change of concavity. For instance, for

   $$f(x) = x^4 + x^2$$
   $$f'(x) = 4x^3 + 2x$$
   $$f''(x) = 12x^2 + 2$$

   there is no c in (a, b) for which $f''(c) = 12x^2 + 2 = 0$.

   CHOICE B

8. The absolute maximum value of the function $f(\theta)$ is the largest function value to occur at a critical number or at an endpoint on $[0, \pi]$.

$$f(\theta) = \cos\theta - \cos^2\theta$$
$$f'(\theta) = -\sin\theta - 2\cos\theta(-\sin\theta) = -\sin\theta + 2\sin\theta\cos\theta$$

To determine any critical number(s),

$$\text{set } f'(\theta) = -\sin\theta + 2\sin\theta\cos\theta = 0$$
$$-\sin\theta(1 - 2\cos\theta) = 0$$
$$\sin\theta = 0 \quad 1 - 2\cos\theta = 0$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = 0 \quad \theta = \pi \quad \theta = \frac{\pi}{3}$$

Evaluate $f(0)$, $f(\pi)$, and $f\left(\frac{\pi}{3}\right)$ to determine the absolute maximum value.

$$f(0) = \cos 0 - \cos^2 0 = 0$$
$$f(\pi) = \cos\pi - \cos^2\pi = -2$$
$$f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} - \cos^2\frac{\pi}{3} = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$\boxed{\text{The absolute maximum value is } \frac{1}{4}}$.

CHOICE C

9.
$$
\begin{aligned}
V &= \pi r^2 h \\
h + 2\pi r &= 300 \\
h &= 300 - 2\pi r \\
V &= \pi r^2(300 - 2\pi r) \\
V &= 300\pi r^2 - 2\pi^2 r^3 \\
V' &= 600\pi r - 6\pi^2 r^2 \\
\text{Let } V' &= 600\pi r - 6\pi^2 r^2 = 0 \\
6\pi r(100 - \pi r) &= 0 \\
100 - \pi r &= 0 \\
r &= \boxed{\frac{100}{\pi}\text{ m}} \\
h &= 300 - 2\pi r \\
&= 300 - 2\pi\left(\frac{100}{\pi}\right) \\
h &= \boxed{100\text{ m}}.
\end{aligned}
$$

CHOICE B

10. $\frac{dy}{dx} = F'(x) = f(x) = 3x^2 + \sec^2 x - 4e^x$

$y = \boxed{x^3 + \tan x - 4e^x + C}$.

CHOICE A

11. (a) For f, a polynomial function, the critical numbers will be determined at

$$
\begin{aligned}
f'(x) &= 0 \\
\text{For } f(x) &= x^3 + 3x^2 + 2 \\
f'(x) &= 3x^2 + 6x \\
\text{Let } f'(x) &= 3x^2 + 6x = 0 \\
3x(x+2) &= 0 \\
x = 0 &\quad x = -2
\end{aligned}
$$

The critical numbers for f are both $\boxed{x = 0 \text{ and } x = -2}$

(b)

Interval	Sign of x	Sign of $x+2$	Sign of $f'(x) = 3x^2 + 6x$	Conclusion
$(-\infty, -2)$	$-$	$-$	$+$	Increasing
$(-2, 0)$	$-$	$+$	$-$	Decreasing
$(0, \infty)$	$+$	$+$	$+$	Increasing

f is increasing on $\boxed{(-\infty, -2] \text{ and on } [0, \infty)}$

(c) From the chart in (b) above, the local extreme points occur at $x = -2$ and $x = 0$

$f(-2) = (-2)^3 + 3(-2)^2 + 2 = 6$

$f(0) = 0^3 + 3(0)^2 + 2 = 2$

By the first derivative test, there is a $\boxed{\text{local maximum at } (-2, 6)}$ since f changes from increasing to decreasing from left to right about the critical number, $x = -2$.

By the first derivative test, there is a $\boxed{\text{local minimum at } (0, 2)}$ since f changes from decreasing to increasing from left to right about the critical number, $x = 0$.

(d) For $f(x) = x^3 + 3x^2 + 2$

$$f'(x) = 3x^2 + 6x$$
$$f''(x) = 6x + 6$$

Let $f''(x) = 6x + 6 = 0$

$$x = -1$$

Interval	Sign of $f''(x)$	Conclusion
$(-\infty, -1)$	$-$	Concave Down
$(-1, \infty)$	$+$	Concave Up

(e) Referring to the chart in (d) above, there is a Point of Inflection at $x = -1$ since f changes concavity, from concave down to concave up, at $x = -1$.

$$f(-1) = (-1)^3 + 3(-1)^2 + 2 = 4$$

The point of inflection is at $\boxed{(-1, 4)}$.

(f) $y - 4 = f'(-1)(x - (-1))$
$y - 4 = -3(x + 1)$
$y - 4 = -3x - 3$
$\boxed{y = -3x + 1}$.

12. (a) The object is at rest when $x'(t) = 0$ which, here, is when the graph of $x(t)$, as provided, has horizontal tangents. Therefore, the object is at rest at both $\boxed{t = 1 \text{ and } t = 5}$.

(b) The velocity of the object is increasing when $x''(t) > 0$ which occurs here when the graph of $x(t)$ is concave up which is on the interval $\boxed{(3, 7)}$.

13. (a) $\dfrac{d^2y}{dx^2} = 3x^2 - 6x$

$$\frac{dy}{dx} = x^3 - 3x^2 + C_1$$

$$\boxed{y = \frac{x^4}{4} - x^3 + C_1x + C_2}.$$

(b) $y = \dfrac{x^4}{4} - x^3 + C_1x + C_2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^4}{4}\right) - \frac{d}{dx}(x^3) + \frac{d}{dx}(C_1x) + \frac{d}{dx}(C_2)$$
$$= x^3 - 3x^2 + C_1$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + \frac{d}{dx}(C_1)$$
$$= 3x^2 - 6x.$$

(c) $y = \frac{x^4}{4} - x^3 + C_1 x + C_2$

For $(0,2), 2 = C_2$

For $(1,3), 3 = \frac{1^4}{4} - 1^3 + C_1(1) + 2$

$3 = \frac{1}{4} - 1 + C_1 + 2$

$C_1 = \frac{7}{4}$

$\boxed{y = \frac{x^4}{4} - x^3 + \frac{7}{4}x + 2}.$

AP® Cumulative Review Problems: Chapters 1–5

1. $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{6}+h\right) - \frac{\sqrt{3}}{2}}{h} = \lim_{h\to 0} \frac{\cos\frac{\pi}{6}\cos h - \sin\frac{\pi}{6}\sin h - \frac{\sqrt{3}}{2}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{3}}{2}\cos h - \frac{1}{2}\sin h - \frac{\sqrt{3}}{2}}{h}$

$= \lim_{h\to 0} \frac{-\frac{\sqrt{3}}{2}(1-\cos h) - \frac{1}{2}\sin h}{h} = -\frac{\sqrt{3}}{2}\lim_{h\to 0}\frac{1-\cos h}{h} - \frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h}$

$= -\frac{\sqrt{3}}{2}(0) - \frac{1}{2}(1) = -\frac{1}{2}$

CHOICE B

2. $V = x^3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ Given $\frac{dx}{dt} = 3$, determine $\frac{dV}{dt}$ at $x = 12$

$\frac{dV}{dt} = 3(12)^2 3 = 864 \text{ cm}^3/\text{min}$

CHOICE C

3. $\lim_{x\to 0}\frac{e^x \sin x \tan x}{x^2} = \lim_{x\to 0}\frac{e^x \sin x\left(\frac{\sin x}{\cos x}\right)}{x^2} = \lim_{x\to 0}\frac{e^x \sin x(\sin x)}{x^2\cos x} = \lim_{x\to 0}\left(\frac{e^x}{\cos x}\right)\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{x}\right)$

$\left(\lim_{x\to 0}\frac{e^x}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = \left(\frac{e^0}{\cos 0}\right)(1)(1) = (1)(1)(1) = 1$

CHOICE B

4. $f(x) = 3x\sqrt{\cos(3x)} = 3x(\cos(3x))^{\frac{1}{2}}$

Use the Product Rule to find $f'(x)$.

Then find $f'(0)$

$f'(x) = 3\left[\left(\frac{d}{dx}x\right)(\cos(3x))^{\frac{1}{2}} + \frac{d}{dx}(\cos(3x))^{\frac{1}{2}}x\right] = 3\left[(\cos(3x))^{\frac{1}{2}} - \frac{1}{2}(\cos(3x))^{-\frac{1}{2}}(3)(x)\right]$

$= 3(\cos(3x))^{\frac{1}{2}} - \frac{9x}{2(\cos(3x))^{\frac{1}{2}}} = \frac{6\cos(3x) - 9x}{2(\cos(3x))^{\frac{1}{2}}}$

$f'(0) = \frac{6\cos 0 - 9(0)}{2(\cos 0)^{\frac{1}{2}}} = \frac{6}{2} = 3$

CHOICE D

5. Find the critical number(s) of $f(x) = 2xe^{-x^2}$ by solving

$f'(x) = 0$ and finding when $f'(x)$ does not exist.

$f(x) = 2xe^{-x^2}$ Find $f'(x)$ by using the Product Rule.

$$f'(x) = 2\left(\frac{d}{dx}x\left(e^{-x^2}\right) + \frac{d}{dx}\left(e^{-x^2}\right)x\right) = 2\left(e^{-x^2} + -2xe^{-x^2}(x)\right) = \frac{2-4x^2}{e^{x^2}}$$

Let $\frac{2-4x^2}{e^{x^2}} = 0 \quad 2 - 4x^2 = 0 \; 4x^2 = 2 \; x^2 = \frac{1}{2} \; x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

Let $e^{x^2} = 0$ There is no solution to this equation. The only critical numbers are $\pm\frac{\sqrt{2}}{2}$

CHOICE C

6. In order to find the intervals on which the function $f(x) = x^3 - x^2 - 8x$ is increasing use the Increasing/Decreasing Function Test which, in pertinent part, states that f is increasing on intervals where $f'(x) > 0$. We determine those intervals by first solving for $f'(x) = 0$ to form three intervals.

$$f(x) = x^3 - x^2 - 8x$$
$$f'(x) = 3x^2 - 2x - 8$$
$$\text{Let } f'(x) = 3x^2 - 2x - 8 = 0$$
$$(3x+4)(x-2) = 0$$
$$x = \frac{-4}{3} \quad x = 2$$

We use the critical numbers $\frac{-4}{3}$ and 2 to form three intervals as shown below. Then we determine the sign of $f'(x)$ in on each interval, as shown below.

Interval	Sign of $3x+4$	Sign of $x-2$	Sign of $f'(x) = 3x^2 - 2x - 8$	Conclusion
$\left(-\infty, \frac{-4}{3}\right)$	Negative (−)	Negative (−)	Positive (+)	f is increasing
$\left(\frac{-4}{3}, 2\right)$	Positive (+)	Negative (−)	Negative (−)	f is decreasing
$(2, \infty)$	Positive (+)	Positive (+)	Positive (+)	f is increasing

We conclude that f is increasing on the intervals $\left(-\infty, \frac{-4}{3}\right)$ and $(2, \infty)$. Since f is continuous on its domain, f is increasing on the intervals $\left(-\infty, \frac{-4}{3}\right)$ and $(2, \infty)$.

CHOICE A

7. $f(x) = \sec^{-1}(3x) \; f'(x) = \frac{\frac{d}{dx}(3x)}{3x\sqrt{(3x)^2-1}} = \frac{3}{3x\sqrt{9x^2-1}} = \frac{1}{x\sqrt{9x^2-1}}$

CHOICE B

8. Since the power of the numerator in $f(x) = \frac{x^2+x-6}{x^4-16}$ is less than the power of the denominator, there is a horizontal asymptote at $y = 0$. Simplify $f(x) = \frac{x^2+x-6}{x^4-16}$.

$$f(x) = \frac{x^2+x-6}{x^4-16} = \frac{(x+3)(x-2)}{(x^2+4)(x+2)(x-2)} = \frac{x+3}{(x^2+4)(x+2)}$$

To find the equation(s) of the vertical asymptote(s), set the denominator of the simplified form of $f(x) = 0$ and solve for the real number that satisfies the equation.

$$x + 2 = 0$$
$$x + 2 = 0$$
$$x = -2$$

There is a vertical asymptote at $x = -2$. Avoid the mistake of setting the denominator of the unsimplified form of $f(x) = 0$ which would have generated an additional value of $x = 2$. There is a hole in the graph at $x = 2$ not a vertical asymptote.

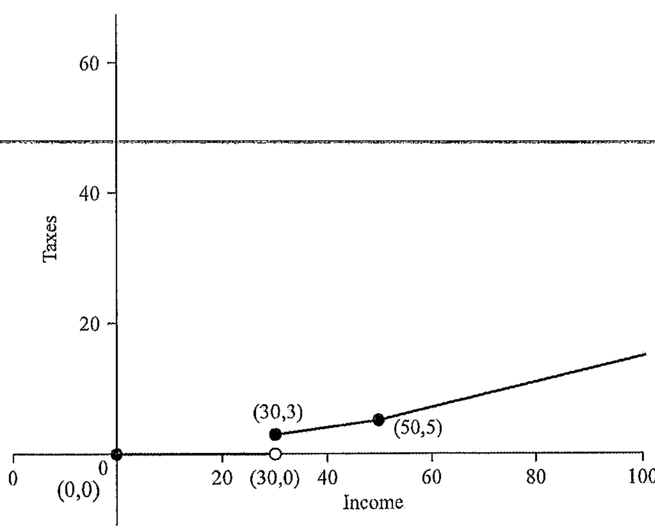
CHOICE D

9.

$$x^2y - 2x = 5$$
$$x^2y - 2x = 5$$
$$2xy + x^2\frac{dy}{dx} - 2 = 0$$
$$x^2\frac{dy}{dx} = 2 - 2xy$$
$$\frac{dy}{dx} = \frac{2-2xy}{x^2}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2-2xy}{x^2}\right) = \frac{\frac{d}{dx}(2-2xy)\,x^2 - (2-2xy)\,\frac{d}{dx}x^2}{x^4}$$
$$= \frac{-2\left(y + x\frac{dy}{dx}\right)x^2 - (2-2xy)(2x)}{x^4} = \frac{-2x^2y - 2x^3\frac{dy}{dx} - 4x + 4x^2y}{x^4}$$
$$= \frac{2x^2y - 4x - 2x^3\frac{dy}{dx}}{x^4} = \frac{x\left(2xy - 4 - 2x^2\frac{dy}{dx}\right)}{x^4} = \frac{2xy - 4 - 2x^2\frac{dy}{dx}}{x^3}$$
$$= \frac{2xy - 4 - 2x^2\left(\frac{2-2xy}{x^2}\right)}{x^3} = \frac{2xy - 4 - 4 + 4xy}{x^3} = \frac{6xy - 8}{x^3}$$

CHOICE A

10. (a) The graph below is in units of 1,000 for both the x and y coordinates.



$$T(i) = \begin{cases} 0, \text{if } 0 \leq i < 30{,}000 \\ 0.10i, \text{if } 30{,}000 \leq i < 50{,}000 \\ 5{,}000 + 0.20\,(i - 50{,}000) = 0.20i - 5{,}000, \text{if } 50{,}000 \leq i \end{cases}$$

(b) The function T is defined on the domain $\{i\,|i \geq 0\}$. T is continuous on the domain except at $i = 30$ for the following reasons:

$$\lim_{i \to 30{,}000^-} T(i) = 0 \quad \lim_{i \to 30{,}000^+} T(i) = 3,000$$

Since $\lim_{i \to 30{,}000^-} T(i) \neq \lim_{i \to 30{,}000+} T(i)$ we conclude that $\lim_{i \to 30{,}000} T(i)$ does not exist.

Since $\lim_{i \to 30{,}000} T(i)$ does not exist T is not continuous at $i = 30,000$.

(c) T is not differentiable at $i = 30,000$ because since T is discontinuous at $i = 30,000$, as stated in part b above, then T is not differentiable at $i = 30,000.00$.

(d) T is not differentiable at $i = 50,000$ because $\lim_{i \to 50{,}000} \frac{T(i)-T(50{,}000)}{i-50{,}000}$ does not exist.

$\lim_{i \to 50{,}000} \frac{T(i)-T(50{,}000)}{i-50{,}000}$ does not exist because

$\lim_{i \to 50{,}000^-} \frac{T(i)-T(50{,}000)}{i-50{,}000} = 0.10$ while $\lim_{i \to 50{,}000^+} \frac{T(i)-T(50{,}000)}{i-50{,}000} = 0.20$.

Since $\lim_{i \to 50{,}000-} \frac{T(i)-T(50{,}000)}{i-50{,}000} \neq \lim_{i \to 50{,}000+} \frac{T(i)-T(50{,}000)}{i-50{,}000}$ we conclude

that $\lim_{i \to 50{,}000} \frac{T(i)-T(50{,}000)}{i-50{,}000}$ does not exist.