Key

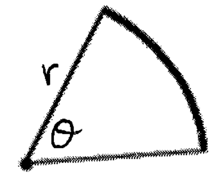
BC Calculus – 9.4b Notes – Area Bounded by a single Polar Curve

9.4b Area Bounded by a Single Polar Curve

sector or slice of a circle

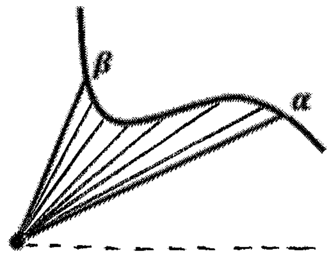
Recall: In geometry, we learned that the area of a sector is $A =$

$A = \frac{1}{2}\theta r^2$



$r = f(\theta)$

radius is some function $f(\theta)$ in terms of θ.



The radius of a sector $= f(\theta_i)$

The central angle $= \frac{\beta - \alpha}{n} = \Delta\theta$

$A \approx \sum_{i=1}^{n} \frac{1}{2}\Delta\theta\left[f(\theta_i)\right]^2$

Push the number of slices up to infinity, and we get

$$A = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{n}\left[f(\theta_i)\right]^2\Delta\theta$$

This is the definition of integration!

Area of a region bounded by a polar graph

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

The trick with polar graphs is to be careful with what interval it takes to trace out the polar graph. Watch what happens with this example.

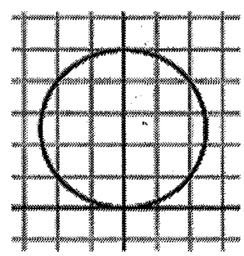
1. Find the area bounded by $r = 5\sin\theta$.

$A = \frac{1}{2}\int_0^{\pi}\left[5\sin\theta\right]^2 d\theta$

$A \approx 19.6349$

$0 = 5\sin\theta$

$\theta = 0, \pi, 2\pi, \ldots$



2. Find the area of the shaded region of the polar curve for $r = 1 - \cos 2\theta$

figure 8 shape (lemniscate)

$r = 1 - \cos 2\theta$

$0 = 1 - \cos(2\theta)$

$\cos 2\theta = 1$

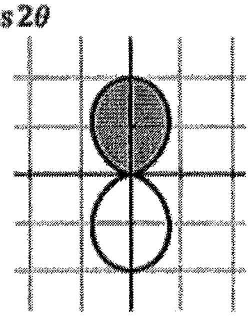
$2\theta = \cos^{-1}(1)$

$2\theta = 0, 2\pi, 4\pi, \ldots$

$\theta = 0, \pi, 2\pi, \ldots$

$A = \frac{1}{2}\int_0^{\pi}\left[1 - \cos 2\theta\right]^2 d\theta$

$\boxed{A \approx 2.356}$



3. Find the area of the inner loop of the limaçon $r = 2\cos\theta + 1$.

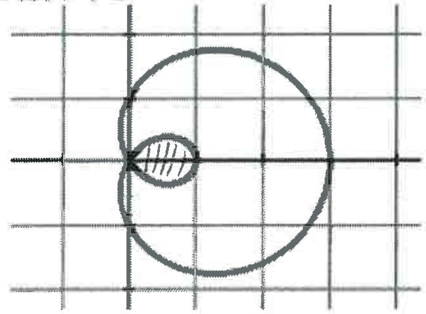
$0 = 2\cos\theta + 1$

$-\frac{1}{2} = \cos\theta$

$\theta = 2\pi/3, 4\pi/3$

$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} [2\cos\theta + 1]^2 d\theta$

$\boxed{A \simeq 0.5435}$



4. Find the area of one petal of the rose curve $r = 3\cos(3\theta)$.

$0 = 3\cos(3\theta)$

$3\cos(3\theta) = 0$

$\cos(3\theta) = 0$

$3\theta = \cos^{-1}(0)$

$\left[3\theta = \pi/2, 3\pi/2, \frac{5\pi}{2}, 7\pi/2\right]\frac{1}{3}$

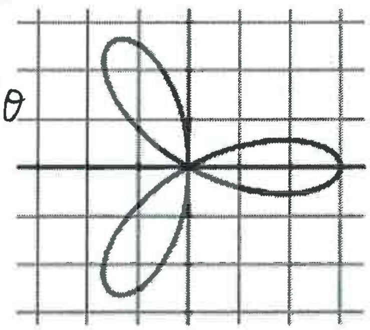
$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, 7\pi/6$

$\downarrow$

$\pi/2$

$A = \frac{1}{2}\int_{\pi/6}^{\pi/2} [3\cos(3\theta)]^2 d\theta$

$\boxed{A \simeq 2.356}$

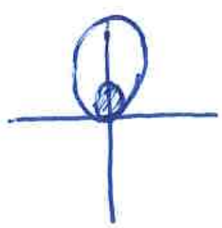


Graph starts at r=4. We want to look and see where else graph hits r=4 again to complete cycle

Find the area of the given region for each polar curve.

to find polar zeros

1. Inside the smaller loop of the limaçon $r = 2\sin\theta + 1$.



$0 = 2\sin\theta + 1$

$\sin\theta = -1/2$

$\theta = \frac{7\pi}{6}, 11\pi/6$

$A = \frac{1}{2}\int_a^b r^2 d\theta$

$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} [2\sin\theta + 1]^2 d\theta$

$\boxed{A \approx 0.5435}$

2. The region enclosed by the cardioid $r = 2 + 2\cos\theta$

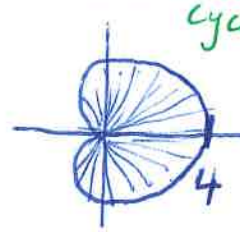
$4 = 2 + 2\cos\theta$

$2\cos\theta = 2$

$\cos\theta = 1$

$\theta = 0, 2\pi$

θ	r
0	4
2π	4



$A = \frac{1}{2}\int_0^{2\pi} [2 + 2\cos\theta]^2 d\theta \simeq \boxed{18.8495}$

*when looking for area of inner loops or petals, set equation equal to 0 to look for polar zeros.

Graph starts at r=6, we want to know where else graph hits r=6 to complete full cycle

3. Inside the graph of the limaçon $r = 4 + 2\cos\theta$.

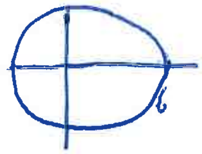
$6 = 4 + 2\cos\theta$

$2\cos\theta = 2$

$\cos\theta = 1$

$\theta = 0, 2\pi$

θ	r
0	6
2π	6



$$A \approx \frac{1}{2}\int_0^{2\pi} [4+2\cos\theta]^2 d\theta \approx \boxed{56.5486}$$

4. Inside one petal of the four-petaled rose $r = \cos 2\theta$.

$0 = \cos 2\theta$

$2\theta = \cos^{-1}(0)$

$2\theta = \pi/2, 3\pi/2, \frac{5\pi}{2}, 7\pi/2$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

$$A = \frac{1}{2}\int_{\pi/4}^{3\pi/4} [\cos(2\theta)]^2 d\theta \approx \boxed{0.3926}$$

5. Inside one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$0 = 4\cos 2\theta$

$0 = \cos 2\theta$

$2\theta = \cos^{-1}(0)$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, 5\pi/2$

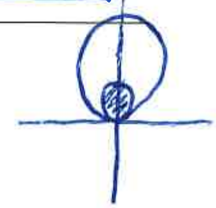
$\theta = \pi/4, 3\pi/4$

$*A = \frac{1}{2}\int_a^b r^2 d\theta$

$r^2 = 4\cos(2\theta)$

$$A = \frac{1}{2}\int_{\pi/4}^{3\pi/4} 4\cos(2\theta)\, d\theta \approx \boxed{2}$$

6. Inside the inner loop of the limaçon $r = 2\sin\theta - 1$.



$0 = 2\sin\theta - 1$

$\sin\theta = 1/2$

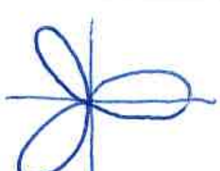
$\theta = \pi/6, 5\pi/6$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2\sin\theta - 1]^2 d\theta \approx \boxed{0.5435}$$

*Draw the 4th quadrant portion of petal from previous cycle $[-2\pi, 0]$ where $(-\pi/6, 0)$ followed by Q1 portion $[0, \pi/6]$

$\frac{11\pi}{6} - \frac{12\pi}{6} = -\pi/6$

7. Write but do not solve, an expression that will give the area enclosed by one petal of the 3 petaled rose $r = 4\cos 3\theta$ found in the first and fourth quadrant.



$0 = 4\cos(3\theta)$

$\cos(3\theta) = 0$

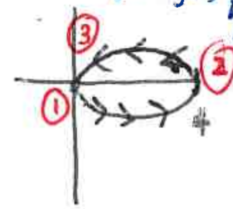
$3\theta = \cos^{-1}(0)$

$3\theta = \pi/2, 3\pi/2, 5\pi/2$

$\theta = \pi/6, 3\pi/6, 5\pi/6, 7\pi/6, \boxed{11\pi/6}$

$$\boxed{A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} [4\cos(3\theta)]^2 d\theta}$$

	θ	r
1	$-\pi/6$	0
2	0	4
3	$\pi/6$	0



8. Write but do not solve an expression that can be used to find the area of the shaded region of the polar curve $r = 3 - 2\sin\theta$.

$1 = 3 - 2\sin\theta$

$2\sin\theta = 2$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

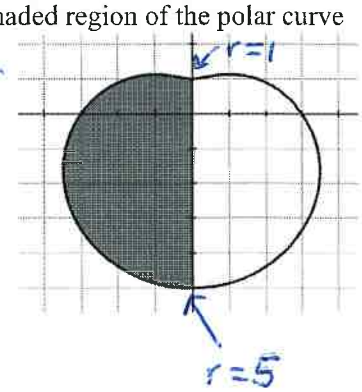
$5 = 3 - 2\sin\theta$

$2\sin\theta = -2$

$\sin\theta = -1$

$\theta = \frac{3\pi}{2}$

$$\boxed{A = \frac{1}{2}\int_{\pi/2}^{3\pi/2} [3 - 2\sin\theta]^2 d\theta}$$



9. Write but do not solve an expression to find the area of the shaded region of the polar curve $r = \cos 2\theta$.

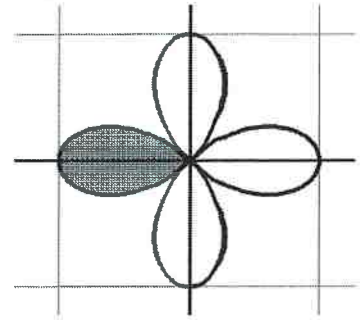
$0 = \cos 2\theta$

$2\theta = \cos^{-1}(0)$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\boxed{A = \frac{1}{2}\int_{3\pi/4}^{5\pi/4} [\cos 2\theta]^2 \, d\theta}$$



10. Find the area of the shaded region of the polar curve $r = 4 - 6\sin\theta$.

$0 = 4 - 6\sin\theta$

$6\sin\theta = 4$

$\sin\theta = 2/3$

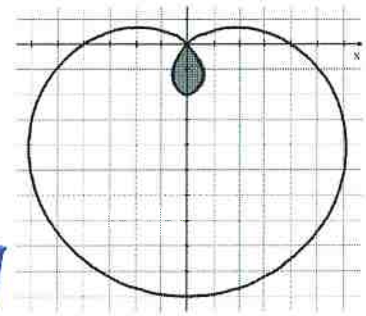
$\theta = \sin^{-1}(2/3)$

$\theta_1 = 0.729727$

*$\sin\theta$ is positive Q1 and Q2

$\theta_2 = \pi - 0.729727 \approx 2.4118649$

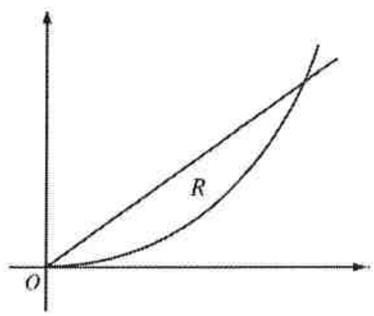
$$A = \frac{1}{2}\int_{0.7297}^{2.4118649} [4 - 6\sin\theta]^2 \, d\theta \approx \boxed{1.7635}$$



Test Prep

Area Bounded by a Polar Curve

11.



Let R be the region in the first quadrant that is bounded by the polar curves $r = \frac{\theta}{2}$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k?

$A = \frac{1}{2}\int_0^k r^2 \, d\theta$

$A = \frac{1}{2}\int_0^k \left[\frac{\theta}{2}\right]^2 d\theta$

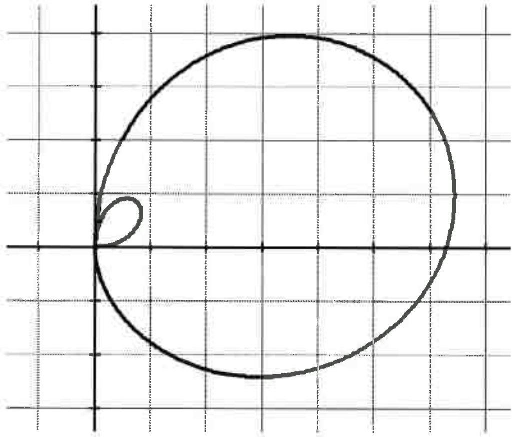
$\frac{1}{2}\int \frac{1}{4}\theta^2 \, d\theta$

$\frac{1}{8}\int \theta^2 \, d\theta$

$\frac{1}{8} \cdot \frac{\theta^3}{3}\Big]_0^k = \frac{1}{24}(k)^3 - \frac{1}{24}(0)$

$= \boxed{\frac{k^3}{24}}$

12.



Calculator active. Consider the polar curve defined by the function $r(\theta) = 2\theta\cos\theta$, where $0 \le \theta \le \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = 2\cos\theta - 2\theta\sin\theta$. The figure above shows the graph of r for $0 \le \theta \le \frac{3\pi}{2}$.

a. Find the area of the region enclosed by the inner loop of the curve.

*find polar zeros

$2\theta\cos\theta = 0$

$2\theta = 0 \quad | \quad \cos\theta = 0$

$\theta = 0 \quad | \quad \theta = \frac{\pi}{2}, 3\pi/2$

$$A = \frac{1}{2}\int_0^{\pi/2} [2\theta\cos\theta]^2 d\theta \approx \boxed{0.5065}$$

b. For $0 \le \theta \le \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

EVT, set $r'(\theta)=0$, find critical points and test endpoints:

$2\cos\theta - 2\theta\sin\theta = 0$

$\theta \approx 0.860336, 3.4256$

θ	r
0	0
0.860	1.122
3.425	-6.5767
$3\pi/2$	0

Greatest distance is 6.5767

c. There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\frac{2}{2-\pi} = \frac{1/2}{dx/d\theta}$$

$$2(dx/d\theta) = \frac{1}{2}(2-\pi)$$

$$\frac{dx}{d\theta} = \frac{1}{2}\left(\frac{1}{2}(2-\pi)\right)$$

$$= \frac{1}{4}(2-\pi) = \boxed{\frac{1}{2} - \frac{\pi}{4}} \text{ or } \boxed{\frac{2-\pi}{4}}$$