Ch. 9 Unit Review AP Practice Problems (p.701-702) – Parametric, Polar & Vector Functions

1. Which of the following parametric equations trace out a circle exactly once?

(A) $x(t) = 2 + \sin t,\ y(t) = 1 + 2\cos t,\ 0 \le t \le 2\pi$

$\boxed{\text{(B)}}$ $x(t) = 2\cos t,\ y(t) = 1 - 2\sin t,\ -\pi \le t \le \pi$

(C) $x(t) = \sin(3t),\ y(t) = \cos(3t),\ 0 \le t \le 2\pi$

(D) $x(t) = \sin t,\ y(t) = \sin t,\ -\pi \le t \le \pi$

$\frac{x}{2} = \cos(t) \quad \frac{y-1}{-2} = \sin(t)$

$\frac{x^2}{4} + \frac{(y-1)^2}{4} = 1$

t	$x = 2\cos(t)$	$y = 1 - 2\sin(t)$
$-\pi$	$x = 2(-1) = -2$	$y = 1 - 2(0) = 1$ ✓
π	$x = 2(-1) = -2$	$y = 1 - 2(0) = 1$ ✓

2. Which parametric equations correspond to the equation $x = 3 - y^2$?

✓ I. $x(t) = 3 - t^2,\ y(t) = t$ ✓

✓ II. $x(t) = 3 - t^{2/3},\ y(t) = t^{1/3}$ $\quad x = 3 - [t^{1/3}]^2 \nearrow x = 3 - y^2$ ✓

III. $x(t) = 3 - t,\ y(t) = t^{1/2}$ only applies for which $y \ge 0$

(A) I only $\quad$ $\boxed{\text{(B)}}$ I and II only

(C) I and III only $\quad$ (D) I, II, and III

3. What is the slope of the tangent line to the curve of the polar equation $r = 2\cos\theta$ when $\theta = \frac{\pi}{3}$?

(A) $-\sqrt{3}$ $\quad$ (B) $\frac{\sqrt{3}}{2}$ $\quad$ $\boxed{\text{(C)}}$ $\frac{\sqrt{3}}{3}$ $\quad$ (D) $-\frac{\sqrt{3}}{3}$

$x = r\cos\theta$

$x = 2\cos\theta \cdot \cos\theta$

$x = 2\cos^2\theta = 2[\cos\theta]^2$

$y = r\sin\theta$

$y = 2\cos\theta\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \rightarrow \frac{-2\sin\theta\sin\theta + 2\cos\theta\cos\theta}{4\cos\theta \cdot -\sin\theta}$

$\frac{dy}{dx} = \frac{-2[\sin\frac{\pi}{3}]^2 + 2[\cos(\frac{\pi}{3})]^2}{-4\cos\frac{\pi}{3}\sin\frac{\pi}{3}} = \frac{-2[\frac{\sqrt{3}}{2}]^2 + 2[\frac{1}{2}]^2}{-4(\frac{1}{2})(\frac{\sqrt{3}}{2})}$

$\frac{-2 \cdot \frac{3}{4} + \frac{2}{4}}{-\sqrt{3}} = \frac{-\frac{6}{4} + \frac{2}{4}}{-\sqrt{3}}$

$\frac{-\frac{4}{4}}{-\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}}$

14. Find an equation of the tangent line to the plane curve represented by the parametric equations $x(t) = 1 + 2\ln t$, $y(t) = t^3 - 3$, at $t = 1$.

$\boxed{\text{(A)}}$ $3x - 2y = 7$ (B) $3x - 2y = 2$

(C) $3x - 2y = -\frac{7}{2}$ (D) $2y + 3x = -\frac{7}{2}$

$\frac{dy}{dt} = 3t^2$ | $\left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1)^2}{\left(\frac{2}{1}\right)} = \frac{3}{2}$ | $y(1) = 1^3 - 3 = -2$ | $\left[y + 2 = \frac{3}{2}(x-1)\right]2$

$\frac{dx}{dt} = 2\left(\frac{1}{t}\right) = \frac{2}{t}$ | $x(1) = 1 + 2\ln 1 = 1$ | point: $(1, -2)$ | $2y + 4 = 3(x-1)$

slope: $m = 3/2$ | $2y + 4 = 3x - 3$

$\boxed{7 = 3x - 2y}$

15. The plane curve C is represented by the parametric equations $x(t) = 2t^2 + 5$, $y(t) = 3t - t^3$. Find all the points on C where the tangent line is horizontal or vertical.

(A) vertical at (5, 0); horizontal at (7, 2)

(B) vertical at (0, 5); horizontal at (7, 2) and (−7, 2)

$\boxed{\text{(C)}}$ vertical at (5, 0); horizontal at (7, 2) and (7, −2)

(D) vertical at (5, 0); horizontal at (7, 2) and (−7, 2)

$\left.\begin{array}{l} x(1) = 2(1)^2 + 5 = 7 \\ y(1) = 3 - 1 = 2 \\ x(-1) = 2(-1)^2 + 5 = 7 \\ y(-1) = 3(-1) - (-1)^3 = -2 \end{array}\right\}$ horizontal

$\left.\begin{array}{l} x(0) = 0 + 5 = 5 \\ y(0) = 0 - 0 = 0 \end{array}\right\}$ vertical

*horizontal tangent occurs where $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 3 - 3t^2$ | $t = 1, -1$ | $\frac{dx}{dt} = 4t$ | $t = 0$

$0 = 3(1 - t^2)$ | $0 = 4t$

16. Find the distance traveled by an object that moves along the plane curve represented by the parametric equations

$x(t) = \frac{3}{4}t^2 + 5$, $y(t) = 3 + t^3$, from $t = 1$ to $t = 4$.

(A) $\frac{65^{3/2}}{8}$ (B) $\frac{60^{3/2}}{8}$

(C) $\frac{65^{3/2} + 5^{3/2}}{8}$ $\boxed{\text{(D)}}$ $\frac{65^{3/2} - 5^{3/2}}{8}$

$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$s = \int_1^4 \sqrt{\left(\frac{3}{2}t\right)^2 + \left(3t^2\right)^2}\, dt$

$\int \sqrt{\frac{9}{4}t^2 + 9t^4}$

$\int \sqrt{\frac{9t^2 + 36t^4}{4}}$

$\int \sqrt{\frac{9t^2(1 + 4t^2)}{4}}$

$\frac{3}{2}\int t\sqrt{1 + 4t^2}\, dt$

$u = 1 + 4t^2$ | $dt = \frac{du}{8t}$

$du/dt = 8t$

$\frac{3}{2}\int \cancel{t} \cdot u^{1/2} \cdot \frac{du}{8\cancel{t}}$

$\frac{3}{16} \cdot \frac{u^{3/2}}{3/2} \rightarrow \frac{3}{16} \cdot \frac{2}{3} u^{3/2}$

$\left.\frac{1}{8}\left(1 + 4t^2\right)^{3/2}\right]_1^4$

$\frac{1}{8}(65)^{3/2} - \frac{1}{8}(5)^{3/2}$

OR

$\frac{65^{3/2} - 5^{3/2}}{8}$

7. The polar equation $r \sin\theta = \frac{5}{4}$ is

$* \; y = r\sin\theta$

$\boxed{y = 5/4}$

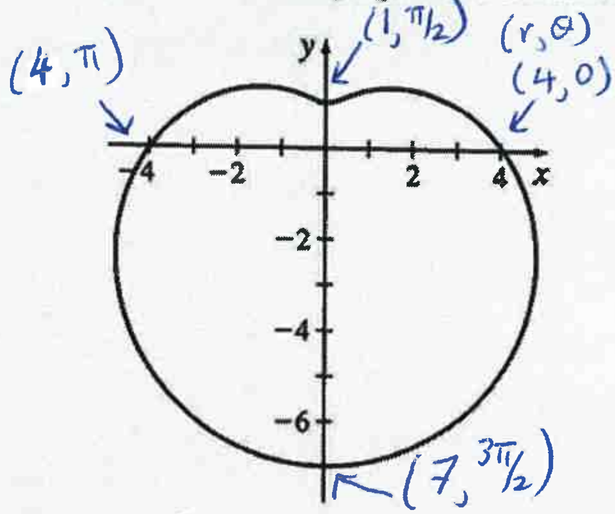
(A) a circle with radius $\frac{\sqrt{5}}{2}$.

(B) a horizontal line $\frac{5}{4}$ units above the polar axis.

(C) a vertical line perpendicular to the polar axis.

(D) a line through the pole with slope $\frac{5}{4}$.

8. Which polar equation has the graph shown below?



θ	$r = 4 - 3\sin\theta$
0	$r = 4 - 3\sin(0) = 4$
$\pi/2$	$r = 4 - 3\sin(\pi/2) = 1$
π	$r = 4 - 3\sin(\pi) = 4$
$3\pi/2$	$r = 4 - 3\sin(3\pi/2) = 4 - 3(-1) = 7$

(A) $r = 4 - 4\sin\theta$ (B) $r = 2 + 3\sin\theta$

(C) $r = 4 - 3\cos\theta$ (D) $r = 4 - 3\sin\theta$

9. Parametric equations for $r = 2^{\theta/3}$ are

(A) $x = 2^{\theta/3}\cos\theta,\ y = 2^{\theta/3}\sin\theta$

(B) $x = r\cos 2^{\theta/3},\ y = r\sin 2^{\theta/3}$

(C) $x = \cos 2^{\theta/3},\ y = \sin 2^{\theta/3}$

(D) $x = \frac{\theta}{3}\cos\theta,\ y = -\frac{\theta}{3}\sin\theta$

$x = r\cos\theta \qquad y = r\sin\theta$

$x = 2^{\theta/3}\cos\theta, \qquad y = 2^{\theta/3}\sin\theta$

10. One petal of the rose $r = \cos(2\theta)$ has perimeter given by the integral

$0 = \cos(2\theta)$
$2\theta = \cos^{-1}(0)$
$2\theta = \pi/2, 3\pi/2$
$\theta = \pi/4, 3\pi/4$

(A) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + \sin^2(\theta)}\, d\theta$

(B) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)}\, d\theta$

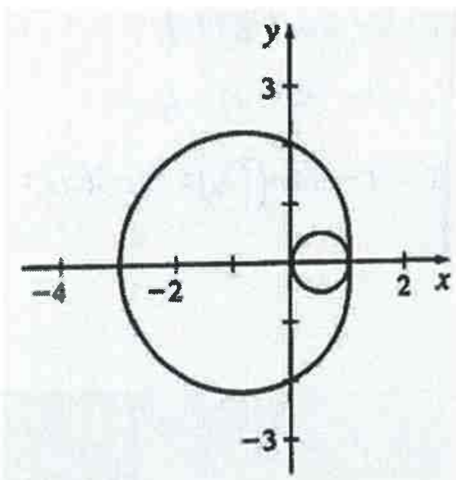
(C) $2\int_0^{\pi/4} \sqrt{\cos^2(2\theta) - 4\sin^2(2\theta)}\, d\theta$

(D) $\int_0^{\pi/4} \sqrt{\cos^2(2\theta) + \sin^2(2\theta)}\, d\theta$

$\frac{dr}{d\theta} = -2\sin(2\theta)$

$S = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \rightarrow \int_0^{\pi/4} \sqrt{[\cos 2\theta]^2 + [-2\sin 2\theta]^2}\, d\theta \rightarrow \boxed{2 \cdot \left[\int_0^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta\right]}$

11. The graphs of the limaçon $r = 2 - \cos\theta$ and the circle $r = \cos\theta$ are shown below.



The area inside the limaçon but outside the circle is given by

I. $\frac{1}{2}\int_0^{2\pi} (2 - \cos\theta)^2 d\theta - \int_0^{\pi/2} \cos^2\theta d\theta$

II. $\frac{1}{2}\int_0^{2\pi} (2 - \cos\theta)^2 d\theta - \frac{\pi}{4}$

Area of circle
πr^2
$\pi\left(\frac{1}{2}\right)^2 = \pi/4$

III. $\frac{1}{2}\int_0^{2\pi} (2 - 2\cos\theta)^2 d\theta$

(A) I only
(B) II only
(C) I and II only
(D) I, II, and III

$c)\ s=\int_0^{\pi/2}\sqrt{(1-\cos t)^2+(\sin t)^2}\,dt$

$=\boxed{4-2\sqrt{2}}$

12. The plane curve represented by
$x(t) = t - \sin t$, $y(t) = 1 - \cos t$
is a cycloid.

$\boxed{\text{SKIP}}$ **(a)** Find the slope of the tangent line to the cycloid ~~for $0 \le t \le 2\pi$.~~ for $t=0$ $\boxed{-3}$

(b) Find an equation of the tangent line to the cycloid at $t = \dfrac{\pi}{3}$. $\boxed{2y-2\sqrt{3}x=\frac{2\pi\sqrt{3}}{3}+4}$

(c) Find the length of the cycloid from $t = 0$ to $t = \dfrac{\pi}{2}$. $4-2\sqrt{2}$

$\frac{dy}{dx}=\frac{-(-\sin t)}{1-\cos(t)}$

$\left.\frac{dy}{dx}\right|_{t=0}=\sin$

$b)\ \left.\frac{dy}{dx}\right|_{t=\pi/3}=\frac{\sin(\pi/3)}{1-\cos(\pi/3)}=\frac{\sqrt{3}/2}{1/2}=\sqrt{3}$

$x(\pi/3)=1-\sqrt{3}/2$

$y(\pi/3)=1-1/2=1/2$

point: $(1-\sqrt{3}/2,\ 1/2)$

slope: $m=\sqrt{3}$

$\boxed{y-1/2=\sqrt{3}(x-(1-\sqrt{3}/2))}$

13. Which of the following is the tangent vector to the curve traced out by the vector function $\mathbf{r}(t) = \langle e^{2t}, 2\sin(3t)\rangle$ at $t = 0$?

(A) $\left\langle \frac{1}{6}, \frac{2}{3} \right\rangle$ (B) $\langle 1, 6\rangle$

$\boxed{\text{(C)}}$ $\langle 2, 6\rangle$ (D) $\langle 2, -6\rangle$

$r'(t)=\langle 2e^{2t},\ 2\cos(3t)\cdot 3\rangle$

$r'(0)=\langle 2e^{0},\ 6\cos(0)\rangle$

$\boxed{r'(0)=\langle 2, 6\rangle}$

14. Determine the domain of the vector function

$$\mathbf{r}(t) = \langle \sqrt{t-1}, -\ln(5-t)\rangle$$

(A) $\{t \mid 1 < t < 5\}$ (B) $\{t \mid t > 1\}$

(C) $\{t \mid t < 5\}$ $\boxed{\text{(D)}}$ $\{t \mid 1 \le t < 5\}$

Domain for $\sqrt{t-1}$ is $t \ge 1$

Domain for $-\ln(5-t)$ is $t<5$

Domain for combined restriction: $\boxed{1\le t<5}$

15. The vector function $\mathbf{r}(t) = \langle 4 - 3t, -t \rangle$ traces out

(A) Two lines: $y = 4 - 3x$ and $y = x$.

(B) Two lines: $x = 4 - 3t$ and $y = -t$.

(C) A line that contains the points (4, 0) and (1, 1).

$\boxed{\text{(D)}}$ A line that contains the points (4, 0) and (1, −1).

$r(0) = (4, 0)$

$r(1) = (1, -1)$

16. The derivative $\mathbf{r}'(t)$ of the vector function $\mathbf{r}(t) = \langle \cos(t^3 + t), \sin(t^3 + t) \rangle$ is

(A) $\langle -\sin(3t^2 + 1), \cos(3t^2 + 1) \rangle$

(B) $\langle (3t^2 + 1)\sin(t^3 + t), (3t^2 + 1)\cos(t^3 + t) \rangle$

(C) $\langle 3t^2 + 1 - \sin(t^3 + t), 3t^2 + 1 + \cos(t^3 + t) \rangle$

$\boxed{\text{(D)}}$ $\langle -(3t^2 + 1)\sin(t^3 + t), (3t^2 + 1)\cos(t^3 + t) \rangle$

$r'(t) = \langle -\sin(t^3+t)\cdot(3t^2+1), \cos(t^3+t)\cdot(3t^2+1) \rangle$

17. A particle moves along the curve traced out by the vector function $\mathbf{r}(t) = \langle 10 \ln t, t^3 + 8 \rangle$, $t > 0$. What is the speed of the object at $t = 2$?

(A) $\frac{5}{12}$ (B) $\frac{12}{5}$ $\boxed{\text{(C)}}$ 13 (D) $\langle 5, 12 \rangle$

speed $= \|r'(2)\| = \sqrt{[10(\frac{1}{t})]^2 + (3t^2)^2} \rightarrow \sqrt{(\frac{10}{2})^2 + (12)^2} = \sqrt{25+144} = \boxed{13}$

18. A particle moves along the curve traced out by the vector function $\mathbf{r}(t) = \langle e^{3t}, 5t - t^2 \rangle$. What is the acceleration of the particle at $t = 0$?

(A) $\langle 9, 0 \rangle$ (B) $\langle 6, -2 \rangle$ $\boxed{\text{(C)}}$ $\langle 9, -2 \rangle$ (D) $\langle 9, 2 \rangle$

$r'(t) = \langle 3e^{3t}, 5-2t \rangle$

$r''(t) = \langle 9e^{3t}, -2 \rangle$

$r''(0) = \langle 9e^0, -2 \rangle \rightarrow \boxed{\langle 9, -2 \rangle}$

19. The arc length of the smooth curve traced out by $\mathbf{r}(t) = \langle e^t, \sin t\rangle$ from $t = 0$ to $t = \pi$ is given by

(A) $\int_0^{\pi} \sqrt{(e^t + \sin t)^2}\, dt$ $\boxed{\text{(B)}}$ $\int_0^{\pi} \sqrt{e^{2t} + \cos^2 t}\, dt$

(C) $\int_0^{\pi} \sqrt{e^{2t} + \sin^2 t}\, dt$ (D) $\int_0^{\pi} \sqrt{(e^t \cos t)^2 + (e^t \sin t)^2}\, dt$

$$s = \int_a^b \sqrt{(x')^2 + (y'(t))^2} \rightarrow \boxed{\int_0^{\pi} \sqrt{(e^t)^2 + (\cos(t))^2}\, dt}$$

20. $\frac{d}{dt}\langle e^{2t}t^3, e^{2t}(4 - 2t)\rangle =$

$\boxed{\text{(A)}}$ $\langle t^2e^{2t}(2t + 3), 2e^{2t}(3 - 2t)\rangle$

(B) $\langle 6t^2e^{2t}, -4te^{2t}\rangle$

(C) $\langle 5t^2e^{2t}, 2e^{2t}\rangle$

(D) $\langle 5t^2e^{2t}(1 + t), 2e^{2t}(3 - 2t)\rangle$

$$\langle e^{2t}(2)t^3 + e^{2t}\cdot 3t^2,\ e^{2t}(2)(4-2t) + e^{2t}(-2)\rangle$$

$$\leftarrow \langle t^2e^{2t}(2t+3),\ 2e^{2t}(4-2t-1)\rangle$$

21. $\int_e^{e^2} \left\langle \frac{3}{t}, 4t \right\rangle dt =$

(A) $\langle 3, 2e^2 - 2e\rangle$ (B) $\langle 3, 2e^4 - e^2\rangle$

(C) $\langle 5, 2e^4 - e^2\rangle$ $\boxed{\text{(D)}}$ $\langle 3, 2e^4 - 2e^2\rangle$

$$3\ln(t)\Big]_e^{e^2} = 3[\ln e^2 - \ln e] = 3(2-1)$$

$$\frac{4t^2}{2}\Big]_e^{e^2} = 2(e^2)^2 - 2e^2 = 2e^4 - 2e^2$$

$$\boxed{\langle 3, 2e^4 - 2e^2\rangle}$$

22. A particle moves along a plane curve with velocity $\mathbf{v}(t) = \langle 3t + 4, \sqrt{t+1}\rangle$. If $\mathbf{r}(0) = \langle 3, 1\rangle$, what is the position of the particle at $t = 3$?

$\boxed{\text{(A)}}$ $\left\langle \frac{57}{2}, \frac{17}{3} \right\rangle$ (B) $\left\langle \frac{57}{2}, \frac{19}{3} \right\rangle$

(C) $\left\langle \frac{57}{2}, 6 \right\rangle$ (D) $\langle 171, 34\rangle$

$$r(t) = \langle \tfrac{3}{2}t^2 + 4t + 3,\ \tfrac{2}{3}(t+1)^{3/2} + \tfrac{1}{3}\rangle$$

$$r(3) = \langle \tfrac{3}{2}(9) + 12 + 3,\ \tfrac{2}{3}(4)^{3/2} + \tfrac{1}{3}\rangle$$

$$\boxed{\langle \tfrac{57}{2}, \tfrac{17}{3}\rangle}$$

$$\int 3t + 4\, dt = \frac{3t^2}{2} + 4t + C_1$$

$$\tfrac{3}{2}(0)^2 + 4(0) + C_1 = 3$$

$$C_1 = 3$$

$$\int (t+1)^{1/2}\, dt = \frac{(t+1)^{3/2}}{3/2} + C_2$$

$$\tfrac{2}{3}(0+1)^{3/2} + C_2 = 1$$

$$C_2 = 1/3$$

23. The solution to the vector differential equation $\mathbf{r}'(t) = \langle 4t,\ 2t^3 \rangle$, given $\mathbf{r}(0) = \langle 1, 2 \rangle$, is

$\boxed{\left\langle \frac{4t^2}{2}+1, \frac{1}{2}t^4 + 2 \right\rangle}$

(A) $\langle 2t^2, t^4 \rangle$ (B) $\left\langle 2t^2 - 1,\ \frac{1}{2}t^4 - 2 \right\rangle$

(C) $\left\langle 2t^2 + 1,\ \frac{1}{2}t^4 + 2 \right\rangle$ (D) $\langle 4t + 1,\ 2t^3 + 2 \rangle$

$\int 4t\,dt = \frac{4t^2}{2} + C_1 \rightarrow \frac{4(0)^2}{2} + C_1 = 1 \rightarrow C_1 = 1$ | $\int 2t^3\,dt = \frac{2t^4}{4} + C_2 \rightarrow \frac{1}{2}(0)^4 + C_2 = 2$

$C_2 = 2$

24. For the vector function $\mathbf{r}(t) = \langle e^t, 2t \rangle$, find the tangent vector to the curve traced out by $\mathbf{r} = \mathbf{r}(t)$ at $t = 0$.

$\boxed{r'(0) = \langle 1, 2 \rangle}$

$r'(t) = \langle e^t, 2 \rangle$

$r'(0) = \langle e^0, 2 \rangle$

25. The position vector of a particle moving in the xy-plane is

$$\mathbf{r}(t) = \langle 2t^3 + 1, 3\cos(2t - 6) \rangle, t \geq 0$$

(a) Find the velocity vector of the particle at $t = 3$. $\langle 54, 0 \rangle$

(b) Find the acceleration vector of the particle at $t = 3$. $\langle 36, -12 \rangle$

(c) Find the speed of the particle at $t = 3$. 54

(d) Write an integral that represents the total distance the particle travels from $t = 0$ to $t = 3$.

$\boxed{6\int_0^3 \sqrt{t^4 + \sin^2(2t-6)}\,dt}$

a) $v(t) = \langle 6t^2, -3\sin(2t-6)\cdot 2 \rangle$

$v(3) = \langle 6(3)^2, -6\sin(0) \rangle \rightarrow \boxed{\langle 54, 0 \rangle}$

b) $a(t) = \langle 12t, -6\cos(2t-6)\cdot 2 \rangle$

$a(3) = \langle 36, -12\cos 0 \rangle \rightarrow \boxed{\langle 36, -12 \rangle}$

c) speed $= \sqrt{54^2 + 0^2} = \boxed{54}$

d) $\boxed{\int_0^3 \sqrt{(6t^2)^2 + (6\sin(2t-6))^2}\,dt}$