Key

Name: Key (WS #1) Date:________ Period: ____ **Review**

①

Unit 9 Review – Parametric Equations, Polar Coordinates, and Vector-Valued Functions (WS #1)

Reviews do NOT cover all material from the lessons but will hopefully remind you of key points. To be prepared, you must study all packets from Unit 9.

1. A curve is defined parametrically by $x(t) = t^3 - 3t^2 + 4$ and $y(t) = \sqrt{t^2 + 16}$. What is the equation of the tangent line at the point defined by $t = 3$?

$x(3) = 3^3 - 3(3)^2 + 4 = 4$

$y(3) = \sqrt{3^2+16} = 5$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \rightarrow \frac{\frac{1}{2}(t^2+16)^{-1/2}(2t)}{3t^2-6t}$

$\left.\frac{dy}{dx}\right|_{t=3} = \frac{y'(3)}{x'(3)} \rightarrow \frac{(3)(3^2+16)^{-1/2}}{3(3)^2-6(3)} \rightarrow \frac{\frac{3}{5}}{9} \rightarrow \frac{3}{5} \cdot \frac{1}{9} = \frac{3}{45} = \frac{1}{15}$

point: $(4, 5)$

slope: $m = 1/15$

$\boxed{y - 5 = \frac{1}{15}(x-4)}$

2. An object moves in the xy-plane so that its position at any time t is given by the parametric equations $x(t) = t^2 + 3$ and $y(t) = t^3 + 5t$. What is the rate of change of y with respect to x when $t = 1$?

$x'(t) = 2t$

$x'(1) = 2$

$y'(t) = 3t^2 + 5$

$y'(1) = 3 + 5 = 8$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8}{2} = \boxed{4}$

since $\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} = \boxed{4}$

3. A curve in the xy-plane is defined by $(x(t), y(t))$, where $x(t) = 3t$ and $y(t) = t^2 + 1$ for $t \geq 0$. What is $\frac{d^2y}{dx^2}$ in terms of t?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t}{3}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \rightarrow \frac{\frac{d}{dx}\left(\frac{2t}{3}\right)}{3} \rightarrow \frac{\frac{2}{3}}{3} \rightarrow \frac{2}{3} \cdot \frac{1}{3} = \boxed{\frac{2}{9}}$

4. If $x(\theta) = \cot\theta$ and $y(\theta) = \csc\theta$, what is $\frac{d^2y}{dx^2}$ in terms of θ?

$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-\csc\theta\cot\theta}{-\csc^2\theta} \rightarrow \frac{\cot\theta}{\csc\theta} \rightarrow \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\csc\theta} \rightarrow \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin\theta}{1} = \cos\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\cos\theta)}{-\csc^2\theta} \rightarrow \frac{-\sin\theta}{-\csc^2\theta} \rightarrow \sin\theta \cdot \sin^2\theta \rightarrow \boxed{\sin^3\theta}$

5. What is the length of the curve defined by the parametric equations $x(t) = 7 + 4t$ and $y(t) = 6 - t$ for the interval $0 \leq t \leq 9$?

$* \text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\int_0^9 \sqrt{(4)^2 + (-1)^2}\, dt$

$\int_0^9 \sqrt{17}\, dt$

$\sqrt{17}t\Big]_0^9 = 9\sqrt{17} - 0\sqrt{17} = \boxed{9\sqrt{17}}$

6. What is the length of the curve defined by the parametric equations $x(\theta) = 3\cos 2\theta$ and $y(\theta) = 3\sin 2\theta$ for the interval $0 \le \theta \le \frac{\pi}{2}$?

$x'(\theta) = -3\sin(2\theta)\cdot 2$

$y'(\theta) = 3\cos(2\theta)\cdot 2$

$\int_0^{\pi/2} \sqrt{(6\sin 2\theta)^2 + (6\cos 2\theta)^2}\, d\theta$

$\int \sqrt{36(\sin^2 2\theta + \cos^2 2\theta)}\, d\theta$

$\int_0^{\pi/2} \sqrt{36(1)}\, d\theta \rightarrow \int 6\, d\theta$

$6\theta \Big]_0^{\pi/2} = 6(\pi/2) - 6(0)$

$= \boxed{3\pi}$

7. If f is a vector-valued function defined by $\langle 2t^3 + 3t^2 + 4t + 1,\ t^3 - 4t - 1\rangle$ then $f''(2) =$

$f'(t) = \langle 6t^2 + 6t + 4,\ 3t^2 - 4\rangle$

$f''(t) = \langle 12t + 6,\ 6t\rangle$

$f''(2) = \langle 24 + 6,\ 12\rangle$

$\boxed{f''(2) = \langle 30, 12\rangle}$

8. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle e^t \sin 3t,\ e^t \cos 3t\rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{6}$.

$f'(t) = \frac{y'(t)}{x'(t)} \rightarrow \frac{e^t\cos(3t) - e^t\sin(3t)\cdot 3}{e^t\sin(3t) + e^t\cdot\cos(3t)\cdot 3} \rightarrow \frac{\cancel{e^t}(\cos 3t - 3\sin 3t)}{\cancel{e^t}(\sin 3t + 3\cos(3t))}$

$f'(\pi/6) = \frac{\cos(3\cdot\pi/6) - 3\sin(3\cdot\pi/6)}{\sin(3\cdot\pi/6) + 3\cos(3\cdot\pi/6)} \rightarrow \frac{\cos(\pi/2) - 3\sin(\pi/2)}{\sin(\pi/2) + 3\cos(\pi/2)} \rightarrow \frac{0-3}{1+3(0)} \rightarrow \frac{-3}{1} \rightarrow \boxed{-3}$

9. Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(0) = \langle -2, 5\rangle$ and $f'(t) = \langle 10t^4,\ 2t\rangle$.

$x(t) = \int 10t^4\, dt = \frac{10t^5}{5} + C$

$x(0) = 2(0)^5 + C$

$-2 = C_1$

$x(t) = 2t^5 - 2$

$y(t) = \int 2t\, dt = \frac{2t^2}{2} + C$

$y(0) = (0)^2 + C$

$5 = C_2$

$y(t) = t^2 + 5$

$\boxed{f(t) = \langle 2t^5 - 2,\ t^2 + 5\rangle}$

10. **Calculator active** For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$ the particle is at position $(3, 4)$. It is known that $\frac{dx}{dt} = \sin 2t$ and $\frac{dy}{dt} = \frac{\sqrt{t}}{e^{2t}}$. Find the y-coordinate of the particles position at time $t = 3$.

$y(1) = 4$

final pos. = initial pos. + displacement

$y(3) = y(1) + \int_1^3 y'(t)\, dt$

$y(3) = 4 + \int_1^3 \frac{\sqrt{t}}{e^{2t}}\, dt$

$y(3) = 4 + 0.0796$

$\boxed{y(3) = 4.0796}$

11. A particle moving in the xy-plane has position given by parametric equations $x(t) = t$ and $y(t) = 4 - t^2$.

A. Find the velocity vector.

$$\boxed{\langle 1, -2t \rangle}$$

B. Find the speed when $t = 1$.

*speed is $\sqrt{[x'(t)]^2+[y'(t)]^2} \rightarrow \sqrt{[x'(1)]^2+[y'(1)]^2} = \sqrt{(1)^2+(-2)^2} = \boxed{\sqrt{5}}$

C. Find the acceleration vector.

$$\boxed{\langle 0, -2 \rangle}$$

12. It is known the acceleration vector for a particle moving in the xy-plane is given by $a(t) = \langle t, \sin t \rangle$. When $t = 0$, the velocity vector $v(0) = \langle 0, -1 \rangle$ and the position vector $p(0) = \langle 0, 0 \rangle$. Find the position vector at time $t = 2$.

$x'(t) = \int t\,dt = \frac{t^2}{2} + c$

$x'(0) = 0$ $\quad 0 = \frac{0^2}{2} + c \quad \underline{c = 0}$

$x'(t) = \frac{1}{2}t^2$

$y'(t) = \int \sin t\,dt = -\cos(t) + c$

$y'(0) = -1 \quad -1 = -\cos(0) + c$

$-1 = -1 + c$

$0 = c$

$y'(t) = -\cos(t)$

$x(t) = \int \frac{1}{2}t^2\,dt = \frac{1}{2}\left(\frac{t^3}{3}\right) + c$

$x(0) = 0 \quad 0 = \frac{1}{6}(0)^3 + c \quad \underline{c = 0}$

$y(t) = \int -\cos(t)\,dt = -\sin(t) + c$

$y(0) = 0 \quad 0 = -\sin(0) + c$

$c = 0$

$p(t) = \langle \frac{1}{6}t^3, -\sin(t) \rangle$

$p(2) = \langle \frac{1}{6}(8), -\sin(2) \rangle$

$\boxed{p(2) = \langle \frac{4}{3}, -\sin(2) \rangle}$

13. Find the slope of the tangent line to the polar curve $r = 2\cos 4\theta$ at the point where $\theta = \frac{\pi}{4}$.

$x(\theta) = r\cos\theta$

$x(\theta) = 2\cos(4\theta)\cdot\cos\theta$

$x'(\theta) = -2\sin 4\theta\cdot 4\cdot\cos\theta + 2\cos(4\theta)\cdot -\sin\theta$

$x'(\theta) = -8\sin 4\theta\cos\theta - 2\cos 4\theta\sin\theta$

$x'(\pi/4) = -8\sin(\pi)\cos(\pi/4) - 2\cos(\pi)\sin(\pi/4)$

$= 0(\frac{\sqrt{2}}{2}) - 2(-1)(\frac{\sqrt{2}}{2}) = \sqrt{2}$

$y(\theta) = r\sin\theta$

$y(\theta) = 2\cos 4\theta\cdot\sin\theta$

$y'(\theta) = -2\sin 4\theta\cdot 4\cdot\sin\theta + 2\cos 4\theta\cdot\cos\theta$

$y'(\pi/4) = -8\sin(\pi)\sin(\pi/4) + 2\cos(\pi)\cos(\pi/4)$

$= 0 + 2(-1)(\frac{\sqrt{2}}{2}) = -\sqrt{2}$

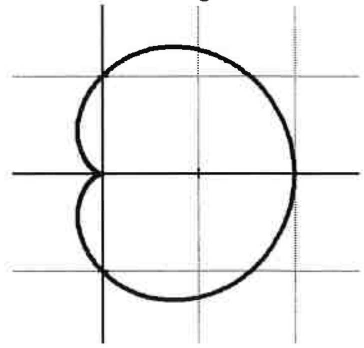
slope $= \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{-\sqrt{2}}{\sqrt{2}} = \boxed{-1}$

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 6$?

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}\left[\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right]}{\cos\theta - \theta\sin\theta} \text{ at } \theta = 6 \rightarrow \frac{5.466085}{2.636663}$$

$$\boxed{2.073}$$

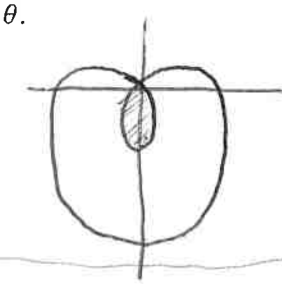
15. **Calculator active.** Find the total area enclosed by the polar curve $r = 1 + \cos\theta$ shown in the figure above.



$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[1+\cos\theta\right]^2 d\theta = \boxed{4.712}$$

16. **Calculator active.** Find the area of the inner loop of the polar curve $r = 3 - 6\sin\theta$.



$0 = 3 - 6\sin\theta$

$6\sin\theta = 3$

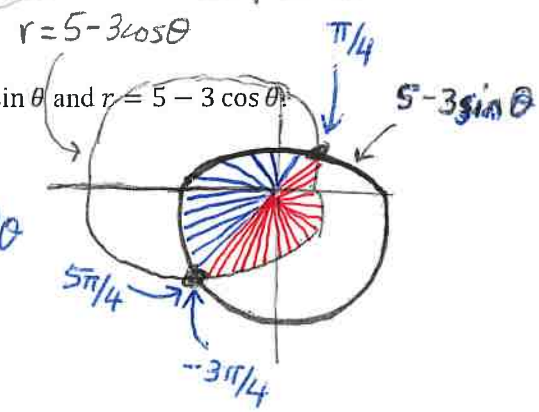
$\sin\theta = 1/2$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[3-6\sin\theta\right]^2 d\theta$$

$$\boxed{4.8916}$$

17. Find the total area of the common interior of the polar graphs $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$.



$5 - 3\sin\theta = 5 - 3\cos\theta$

$-3\sin\theta = -3\cos\theta$

$\sin\theta = \cos\theta$

or

$\tan\theta = 1$

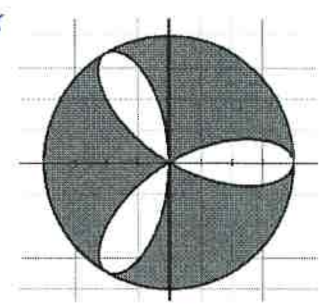
$\theta = \pi/4, 5\pi/4$

(Intersections of both graphs)

$$\int_{-3\pi/4}^{\pi/4} \frac{1}{2}\left[5-3\cos\theta\right]^2 d\theta + \int_{\pi/4}^{5\pi/4} \frac{1}{2}\left[5-3\sin\theta\right]^2 d\theta$$

$$= \boxed{50.2505}$$

18. **Calculator active.** The figure shows the graphs of the polar curves $r = 4\cos 3\theta$ and $r = 4$. What is the sum of the areas of the shaded regions?



*rose curve completes one cycle in $[0, \pi]$

circle − rose curve

$$\int_0^{2\pi} \frac{1}{2}[4]^2 - \int_0^{\pi} \frac{1}{2}\left[4\cos(3\theta)\right]^2 d\theta$$

$$= \boxed{37.699}$$

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BC Calculus Unit 9 Parametric & Polar Test Review WS #2

Calculators Allowed: Show all work that lead to your answer to earn full credit.

1. What is the length of the curved defined by the parametric equations $x(t) = 9\cos t$ and $y(t) = 9\sin t$ for the interval $0 \le t \le 2\pi$?

$\frac{dx}{dt} = -9\sin(t)$

$\frac{dy}{dt} = 9\cos(t)$

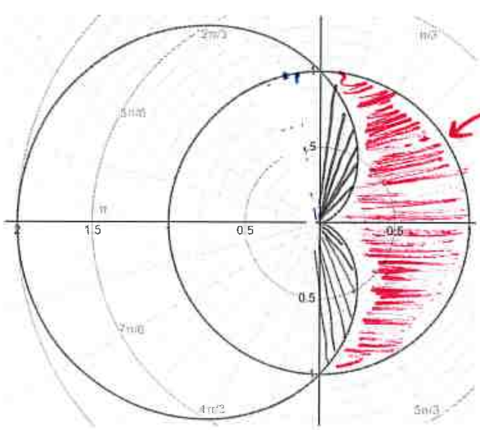
$L = \int_0^{2\pi} \sqrt{(-9\sin t)^2 + (9\cos t)^2}\,dt$

$= \int_0^{2\pi} \sqrt{81\sin^2 t + 81\cos^2 t}\,dt$

$\int \sqrt{81(\sin^2 t + \cos^2 t)}\,dt \rightarrow \int 9\sqrt{\sin^2 t + \cos^2 t}\,dt$

$9\int \sqrt{1}\,dt \rightarrow 9t\Big]_0^{2\pi} = 9(2\pi) - 0 = \boxed{18\pi}$

2. **Calculator active.** Find the area of the region inside the circle $r = 1$ and outside the cardiod $r = 1 - \cos\theta$.



Area of Region = Semicircle − 2(half-loop)

$= \frac{1}{2}\pi r^2 - 2\left[\frac{1}{2}\int_0^{\pi/2} [1-\cos\theta]^2\,d\theta\right]$

$= \frac{\pi}{2}(1)^2 - 0.35619$

$\approx \boxed{1.215}$

3. If $x(t) = 2t^3$ and $y(t) = t^3 - t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{3t^2 - 1}{6t^2} \rightarrow \frac{3t^2}{6t^2} - \frac{1}{6t^2} \rightarrow \frac{1}{2} - \frac{1}{6}t^{-2}$

$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'}{dx/dt} \rightarrow \frac{0 + 2\cdot\frac{1}{6}t^{-3}}{6t^2} \rightarrow \frac{\frac{1}{3t^3}}{6t^2} \rightarrow \frac{1}{3t^3}\cdot\frac{1}{6t^2} = \boxed{\frac{1}{18t^5}}$

4. The position of a remote-controlled vehicle moving along a flat surface at time t is given by $(x(t), y(t))$, with velocity vector $v(t) = \langle 3t^2, 2t\rangle$ for $0 \le t \le 3$. Both $x(t)$ and $y(t)$ are measured in meters, and time t is in seconds. When $t = 0$, the remote-controlled vehicle is at the point $(1, 2)$.

a. Find the acceleration vector of the remote-controlled vehicle when $t = 2$.

$a(t) = \langle 6t, 2\rangle \quad \boxed{a(2) = \langle 12, 2\rangle}$

b. Find the position of the remote-controlled vehicle when $t = 3$.

$x = \int 3t^2\,dt$

$x = \frac{3t^3}{3} + C_1$

$x = t^3 + C_1$

$1 = 0^3 + C_1$

$1 = C_1$

$y = \int 2t\,dt$

$y = \frac{2t^2}{2} + C_2$

$y = t^2 + C_2$

$2 = 0 + C_2$

$2 = C_2$

$x(t) = \langle t^3 + 1, t^2 + 2\rangle$

$x(3) = \langle 27 + 1, 3^2 + 2\rangle$

$\boxed{x(3) = \langle 28, 11\rangle}$

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5. Which of the following gives the length of the path described by the parametric equations $x = 2e^{3t}$ and $y = 3t^2 + t$ from $0 \le t \le 1$?

A. $\int_0^1 \sqrt{12e^{6t} + (6t+1)^2}\,dt$

B. $\int_0^1 \sqrt{4e^{6t} + (6t+1)^2}\,dt$

C. $\int_0^1 \sqrt{4e^{6t} + 9t^4 + t^2}\,dt$

D. $\boxed{\int_0^1 \sqrt{36e^{6t} + (6t+1)^2}\,dt}$

$$*L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dx}{dt} = 2e^{3t} \cdot 3 = 6e^{3t} \qquad \frac{dy}{dt} = 6t+1$$

$$L = \int_0^1 \sqrt{(6e^{3t})^2 + (6t+1)^2}\,dt$$

6. **Calculator active.** A polar curve is given by $r = \frac{5}{3-\sin\theta}$. What angle θ corresponds on the curve with a y-coordinate of -1?

$[0, 2\pi]$

$*y = r\sin\theta$

$$y = \frac{5}{3-\sin\theta} \cdot \sin\theta$$

$$\frac{-1}{1} = \frac{5\sin\theta}{3-\sin\theta}$$

$$5\sin\theta = -1(3-\sin\theta)$$

$$5\sin\theta = -3 + \sin\theta$$

$$4\sin\theta = -3$$

$$\sin\theta = -3/4$$

$$\theta = \sin^{-1}(-3/4)$$

$$\theta = -0.848$$

0.848 0.848

*$\sin\theta$ is negative in Q3, Q4

Q3 $= 0.848 + \pi$

Q4: $2\pi - 0.848$

$$\boxed{\theta = 3.990 \text{ or } \theta = 5.435}$$

7. If f is a vector-valued function defined by $\langle te^t,\ 2t^2e^t\rangle$ then $f''(1) = ?$

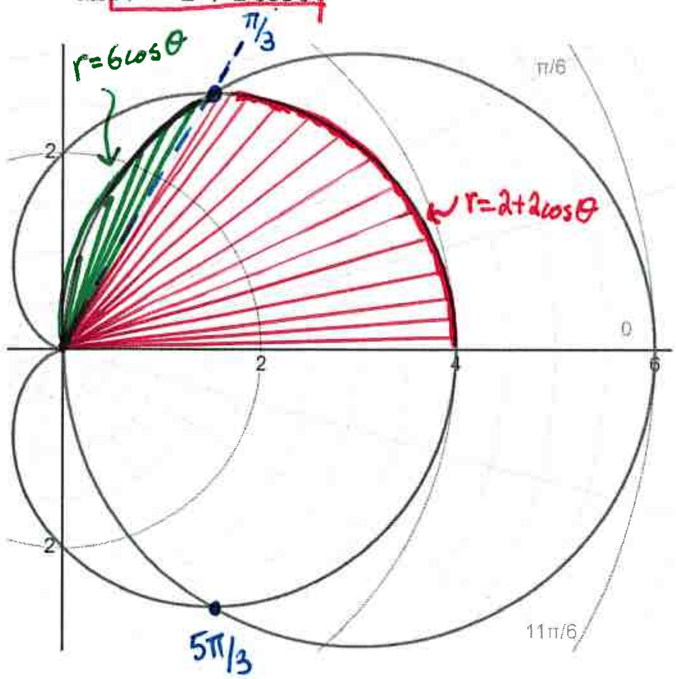
$$f'(t) = \langle 1e^t + te^t,\ 4te^t + 2t^2e^t \rangle$$

$$f''(t) = \langle e^t + 1e^t + te^t,\ 4e^t + 4te^t + 4te^t + 2t^2e^t \rangle$$

$$f''(1) = \langle e + e + e,\ 4e + 4e + 4e + 2e \rangle$$

$$\boxed{f''(1) = \langle 3e, 14e \rangle}$$

8. **Calculator active.** Find the area of the region common to the two regions bounded by the curves $\boxed{r = 6\cos\theta}$ and $\boxed{r = 2 + 2\cos\theta}$.



*find intersection:

$$6\cos\theta = 2 + 2\cos\theta$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pi/3,\ 5\pi/3$$

$$2\left[\frac{1}{2}\int_0^{\pi/3} (2+2\cos\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (6\cos\theta)^2\,d\theta\right]$$

$$14.0774 + 1.6305$$

$$= \boxed{15.708}$$

9. Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(0) = \langle 3, 0\rangle$, and $f'(t) = \langle 4\sin\frac{t}{2}, -2\cos 2t\rangle$.

$x(t) = \int 4\sin\left(\frac{1}{2}t\right)dt$

$u = \frac{1}{2}t$, $\frac{du}{dt} = \frac{1}{2}$, $dt = 2du$

$\int 4\sin u \cdot 2du$

$\int 8\sin u\, du$

$x(t) = -8\cos\left(\frac{t}{2}\right) + C$ ← plug in (0,3)

$3 = -8\cos(0) + C$

$3 = -8 + C$

$-11 = C$

$x(t) = -8\cos(t/2) - 11$

$y(t) = \int -2\cos 2t\, dt$

$u = 2t$, $\frac{du}{dt} = 2$, $dt = \frac{du}{2}$

$y(t) = \int -2\cos u \cdot \frac{du}{2}$

$y(t) = -\sin(2t) + C$ ← plug in (0,0)

$0 = 0 + C$ $\quad C = 0$

$\boxed{f(t) = \left\langle -8\cos\left(\frac{t}{2}\right) + 11, -\sin(2t)\right\rangle}$

10. If $x = 7\cos\theta$ and $y = 7\sin\theta$, find the slope and the concavity at $\theta = \frac{\pi}{4}$.

$y'(\theta) = 7\cos\theta$

$x'(\theta) = -7\sin\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{7\cos(\pi/4)}{-7\sin(\pi/4)} = -1$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}[-\cot\theta]}{-7\sin\theta} \rightarrow \frac{+\csc^2(\theta)}{-7\sin\theta} = -\frac{1}{7}\csc^3\theta$

$\frac{dy}{dx} = \cot\theta$

$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = -\frac{1}{7}\left[\csc(\pi/4)\right]^3 \rightarrow -\frac{1}{7}\cdot\left(\frac{2}{\sqrt{2}}\right)^3 = \boxed{-\frac{2\sqrt{2}}{7}}$ $\boxed{\text{concave down}}$

11. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 9t^2, e^t\rangle$. If the particle is at point $(3, 4)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

*final position = initial position + displacement

$x(b) = x(a) + \int_a^b v(t)dt$

$x(2) = x(0) + \int_0^2 x'(t)dt$

$x(2) = 3 + \int_0^2 9t^2 dt$

$= 3 + 24$

$x(2) = 27$

$y(2) = y(0) + \int_0^2 y'(t)dt$

$y(2) = 4 + \int_0^2 e^t dt$

$y(2) = 4 + 6.389$

$y(2) = 10.389$

At $t = 2$, particle is at point $(27, 10.389)$

Distance from origin is $d = \sqrt{(27-0)^2 + (10.389-0)^2}$

$\boxed{d = 28.930}$

12. Find the slope of the tangent line to the polar curve $r = 2\cos\theta - 1$ at the point where $\theta = \frac{3\pi}{2}$.

$x(\theta) = r\cos\theta$

$x(\theta) = (2\cos\theta - 1)\cos\theta$

$x'(\theta) = (-2\sin\theta)\cos\theta + (2\cos\theta - 1)(-\sin\theta)$

$x'\left(\frac{3\pi}{2}\right) = \left(-2\sin\frac{3\pi}{2}\right)\cos(3\pi/2) + \left(2\cos\frac{3\pi}{2} - 1\right)\left(-\sin\frac{3\pi}{2}\right)$

$= (2)(0) + (0-1)(1)$

$x'\left(\frac{3\pi}{2}\right) = -1$

$y(\theta) = r\sin\theta$

$y(\theta) = (2\cos\theta - 1)\sin\theta$

$y'(\theta) = (-2\sin\theta)\sin\theta + (2\cos\theta - 1)(\cos\theta)$

$y'\left(\frac{3\pi}{2}\right) = -2\left[\sin\left(\frac{3\pi}{2}\right)\right]^2 + \left(2\cos\frac{3\pi}{2} - 1\right)\left(\cos\frac{3\pi}{2}\right)$

$= -2(-1)^2 + (0-1)(0)$

$y'\left(\frac{3\pi}{2}\right) = -2$

$\text{slope} = \frac{y'(3\pi/2)}{x'(3\pi/2)} = \frac{-2}{-1} = \boxed{2}$

13. Find the slope of the tangent line to the curve defined parametrically by $x(t) = 2\cos t$ and $y(t) = 3\sin^2 t$ at $t = \frac{\pi}{3}$.

$y(t) = 3[\sin(t)]^2$

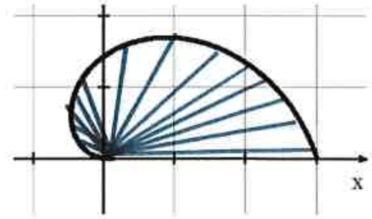
$y'(t) = 6[\sin(t)]\cdot\cos(t)$

$x'(t) = -2\sin(t)$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{6\cancel{\sin t}\cos t}{-2\cancel{\sin t}} = -3\cos(t)$

$\left.\frac{dy}{dt}\right|_{t=\pi/3} = -3\cos(\pi/3) = -3\left(\frac{1}{2}\right) = \boxed{\frac{-3}{2}}$

14. **Calculator active.** The graph shows the polar curve $r = 3 - \theta$ for $0 \leq \theta \leq \pi$. What is the area of the region bounded by the curve and the x-axis?



$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

$$A = \frac{1}{2}\int_0^{\pi} [3-\theta]^2 \, d\theta \approx \boxed{4.500}$$

15. At time t, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle \cos 2t, \sin 4t \rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{4}$.

$$f'(t) = \langle -2\sin(2t), 4\cos(4t) \rangle$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{y'(\pi/4)}{x'(\pi/4)} \rightarrow \frac{4\cos(4\cdot\pi/4)}{-2\sin(2\cdot\pi/4)} \rightarrow \frac{4\cos(\pi)}{-2\sin(\pi/2)} \rightarrow \frac{4(-1)}{-2(1)} \rightarrow \frac{-4}{-2} = \boxed{2}$$

16. Find an equation for the line tangent to the curve given by the parametric equations $x(t) = t^2 + 1$ and $y(t) = t^3 + t + 1$, when $t = 2$.

$x(2) = 2^2 + 1 = 5$

$y(2) = 2^3 + 2 + 1 = 11$

$x'(t) = 2t$

$y'(t) = 3t^2 + 1$

$$\text{slope} = \frac{y'(2)}{x'(2)} = \frac{3(2)^2+1}{2(2)} = \frac{13}{4}$$

point: $(5, 11)$

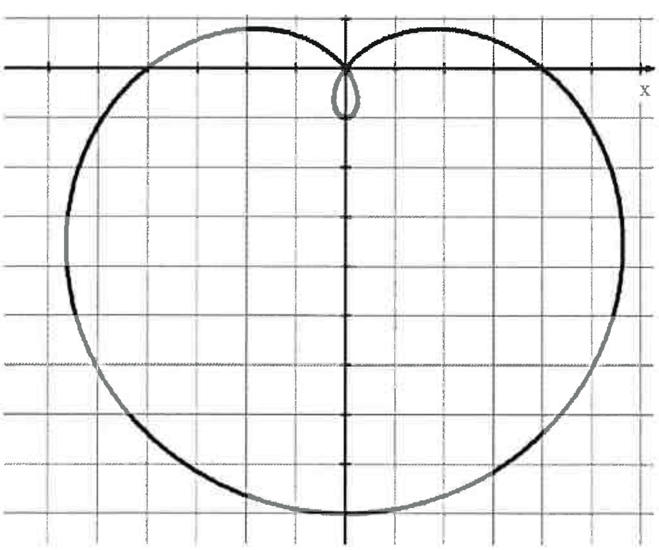
slope: $m = 13/4$

$$\boxed{y - 11 = \frac{13}{4}(x-5) \quad \text{OR} \quad y = \frac{13}{4}x - \frac{21}{4}}$$

17. **Calculator active.** Find the total area enclose by the inner loop of the polar curve $r = 4 - 5\sin\theta$, shown in the figure.

*find polar zeros: $r = 4 - 5\sin\theta$ | $5\sin\theta = 4$ | $\theta \approx 0.927$

$0 = 4 - 5\sin\theta$ | $\sin\theta = 4/5$



*$\sin\theta$ is also positive in 2^{nd} quadrant: since $\theta = 0.927$ is reference angle,

$\theta_2 = \pi - 0.927 \approx 2.214$

$$\text{Area} = \frac{1}{2}\int_{0.927}^{2.214} [4 - 5\sin\theta]^2 \, d\theta \approx \boxed{0.340}$$

(9)

Name: Key Date: ____________ Period: _____

BC Calculus Unit 9 Parametric and Polar Test Review WS #3

Calculators Allowed: Show all work that lead to your answer to earn full credit.

1) What is the slope of the tangent line to the curve defined parametrically by $x(t) = \sqrt{t}$ and $y(t) = \frac{1}{4}(t^2 - 4)$, $t \geq 0$ at the point (2,3)?

$x = \sqrt{t}$
$2 = \sqrt{t}$
$\underline{4 = t}$

$y(t) = \frac{1}{4}(t^2-4)$
$3 = \frac{1}{4}(t^2-4)$
$12 = t^2 - 4$
$16 = t^2$
$t = \pm 4 \rightarrow \underline{t=4}$

$y'(t) = \frac{1}{4} \cdot 2t = \frac{1}{2}t$
$x'(t) = \frac{1}{2}t^{-1/2}$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} \rightarrow t^{3/2}$

$y(t) = \frac{1}{4}t^2 - 1$

$\left.\frac{dy}{dx}\right|_{t=4} = (4)^{3/2} = 2^3 = \boxed{8}$

2) If $x = \sin\theta$ and $y = 2\cos\theta$, what is $\frac{d^2y}{dx^2}$ in terms of θ?

$x'(\theta) = \cos\theta$
$y'(\theta) = -2\sin\theta$

$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-2\sin\theta}{\cos\theta} = -2\tan\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}(-2\tan\theta)}{\cos\theta} \rightarrow \frac{-2\sec^2\theta}{\cos\theta}$

$\boxed{\frac{d^2y}{dx^2} = -2\sec^3\theta}$

3) Which of the following gives the length of the path described by the parametric equations $x = e^{2t}$ and $y = 1 - 2t$ from $0 \leq t \leq 3$?

*Length $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

A. $\int_0^3 \sqrt{4e^{2t} + 4}\,dt$ B. $\int_0^3 \sqrt{2e^{2t} + 2}\,dt$

$\boxed{\text{C. } \int_0^3 \sqrt{4e^{4t} + 4}\,dt}$ D. $\int_0^3 \sqrt{e^{4t} + 4}\,dt$

$\frac{dx}{dt} = 2e^{2t}$ $\frac{dy}{dt} = -2$

$\boxed{L = \int_0^3 \sqrt{(2e^{2t})^2 + (-2)^2}\,dt}$

4) The position of a particle moving in the xy-plane is defined by the vector-valued function, $f(t) = \langle t^3 - 9t^2 + 1,\ 2t^3 - 15t^2 - 36t + 1 \rangle$. For what value of t is the particle at rest?

*set $v(t) = 0$ for both vertical and horizontal components
* find the t value(s) where the 2 components agree.

$f'(t) = \langle 3t^2 - 18t,\ 6t^2 - 30t - 36 \rangle$

$3t^2 - 18t = 0$
$3t(t-6) = 0$
$t = 0,\ \underline{t = 6}$

$6t^2 - 30t - 36 = 0$
$6(t^2 - 5t - 6) = 0$
$6(t-6)(t+1) = 0$
$\underline{t = 6},\ t = -1$

$\boxed{t = 6}$

5) At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle e^{2t}\cos t,\ e^{2t}\sin t\rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

$x'(t) = 2e^{2t}\cos t + e^{2t}(-\sin t)$

$x'(\pi/2) = 2e^{\pi}\cos(\pi/2) + e^{\pi}(-\sin(\pi/2))$

$= 2e^{\pi}(0) + e^{\pi}(-1) = -e^{\pi}$

$y'(t) = 2e^{2t}\sin t + e^{2t}\cdot\cos t$

$y'(\pi/2) = 2e^{\pi}\sin(\pi/2) + e^{\pi}\cos(\pi/2)$

$= 2e^{\pi}(1) + e^{\pi}(0) = 2e^{\pi}$

$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{2e^{\pi}}{-e^{\pi}} = \boxed{-2}$

6) **Calculator active.** At time $t \ge 0$, a particle moving in the xy-plane has a velocity vector given by $v(t) = \langle 2, 2^{-t^2}\rangle$. If the particle is at point $\left(1, \frac{1}{2}\right)$ at time $t = 0$, how far is the particle from the origin at time $t = 1$?

$x(b) = x(a) + \int_a^b x'(t)dt$

$x(1) = x(0) + \int_0^1 2\,dt$

$x(1) = 1 + 2$

$x(1) = 3$

$y(b) = y(a) + \int_a^b y'(t)dt$

$y(1) = y(0) + \int_0^1 2^{-t^2}dt$

$y(1) = 1/2 + 0.8100$

$y(1) = 1.310$

At $t=1$, particle is at point $(1, 1.31)$

Distance from origin is

$d = \sqrt{(3-0)^2 + (1.31-0)^2}$

$\boxed{d = 3.274}$

7) **Calculator active.** The position of a particle at time $t \ge 0$ is given by $x(t) = \frac{\sqrt{t+1}}{3}$ and $y(t) = t^2 + 1$. Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

*Total Distance $= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$x(t) = \frac{1}{3}(t+1)^{1/2}$

$x'(t) = \frac{1}{3}\cdot\frac{1}{2}(t+1)^{-1/2}$

$x'(t) = \frac{1}{6\sqrt{t+1}}$

$y'(t) = 2t$

$\int_0^2 \sqrt{\left(\frac{1}{6\sqrt{t+1}}\right)^2 + (2t)^2}\,dt$

$= \boxed{4.023}$

8) **Calculator active.** The velocity vector a particle moving in the xy-plane has components given by $\frac{dx}{dt} = \sin 2t$ and $\frac{dy}{dt} = e^{\cos t}$. At time $t = 2$, the position of the particle is $(3, 2)$. What is the x-coordinate of the position vector at time $t = 3$?

$x(b) = x(a) + \int_a^b x'(t)dt$

$x(3) = x(2) + \int_2^3 \sin(2t)dt$

$x(3) = 3 + -0.8069$

$\boxed{x(3) = 2.193}$

9) A particle moves along the polar curve $r = 4 - 2\cos\theta$ so that $\frac{d\theta}{dt} = 4$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$.

*think related rates steps

$r = 4 - 2\cos\theta$

$\frac{dr}{dt} = -2(-\sin\theta)\left(\frac{d\theta}{dt}\right)$

$\frac{dr}{dt} = 2\sin\left(\frac{\pi}{3}\right) \cdot 4$

$\frac{dr}{dt} = 8\sin(\pi/3)$

$\frac{dr}{dt} = 8\left(\frac{\sqrt{3}}{2}\right)$

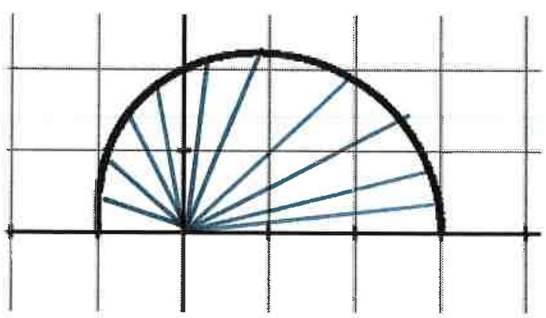
$\boxed{\frac{dr}{dt} = 4\sqrt{3}}$

10) **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = 3\cos\theta - 3\theta\sin\theta$ and $\frac{dy}{d\theta} = 3(\sin\theta + \theta\cos\theta)$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 3$?

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}\left[\frac{3\sin\theta + 3\theta\cos\theta}{3\cos\theta - 3\theta\sin\theta}\right]}{3\cos\theta - 3\theta\sin\theta} \quad \text{at } \theta = 3 \rightarrow \frac{5.5067}{8.0632}$$

$\rightarrow \boxed{0.6829}$

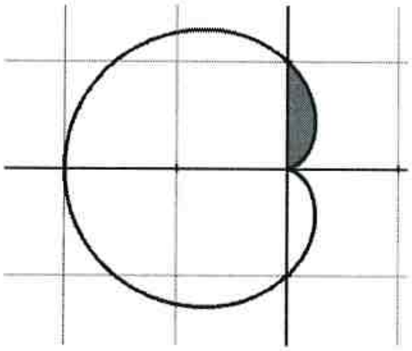
11) The graph to the right shows the polar curve $r = 2 + \cos\theta$ for $0 \le \theta \le \pi$. What is the area of the region bounded by the curve and the x-axis? 7.069



$A = \frac{1}{2}\int_0^{\pi} (2+\cos\theta)^2 \, d\theta$

Area $\approx \boxed{7.069}$

12) Find the area of the shaded region for the polar curve $r = 1 - \cos\theta$. 0.178

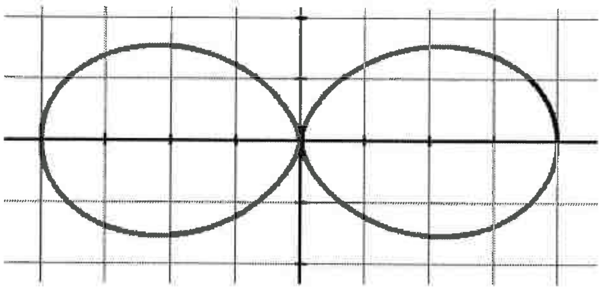


θ	r
0	0
$\pi/2$	1

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2}[1-\cos\theta]^2\,d\theta$$

$$\boxed{\text{Area} \approx 0.178}$$

13) Find the total area enclosed by the polar curve $r = 2 + 2\cos 2\theta$ shown in the figure 18.850

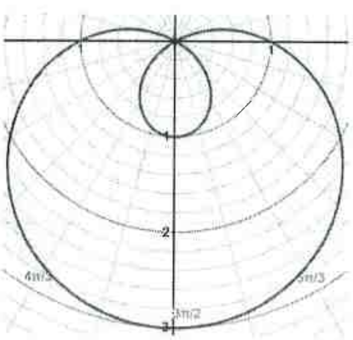


$$A = 2\left[\frac{1}{2}\int_0^{\pi}[2+2\cos(2\theta)]^2\,d\theta\right]$$

OR

$$A = \frac{1}{2}\int_0^{2\pi}(2+2\cos 2\theta)^2\,d\theta \approx \boxed{18.850}$$

14) Write do not solve, an integral expression that represents the area enclosed by the smaller loop of the polar curve $r = 1 - 2\sin\theta$.



* find polar zeros

$r = 1 - 2\sin\theta$

$0 = 1 - 2\sin\theta$

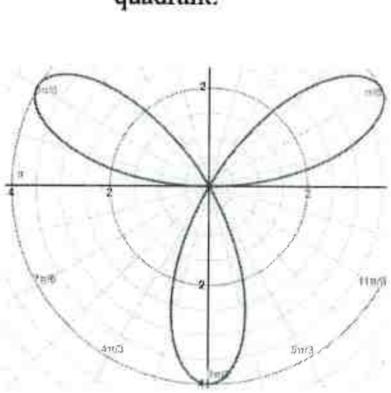
$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6, 5\pi/6$

$$\boxed{\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6}(1-2\sin\theta)^2\,d\theta}$$

15) Find the limits of integration required to find the area of one petal of the polar graph $r = 4\sin 3\theta$ in the second quadrant.



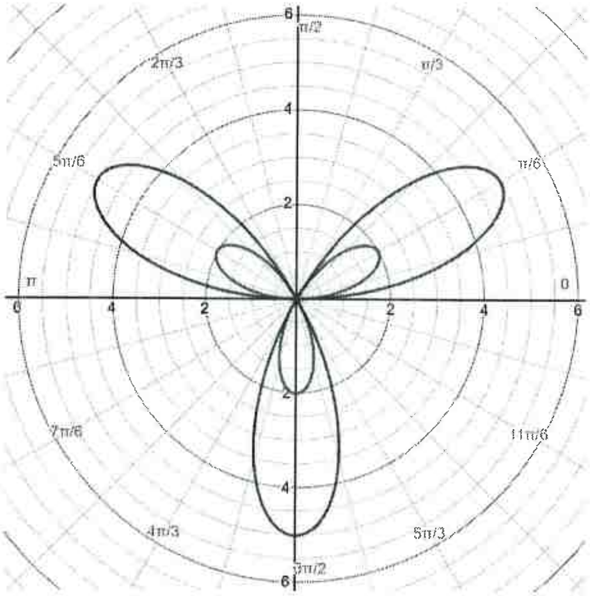
r	θ
$2\pi/3$	$4\sin(3\cdot\frac{2\pi}{3}) = 0$
π	$4\sin(3\pi) = 0$

$$\boxed{\frac{2\pi}{3}, \pi}$$

13

*polar zeros that form first quadrant petal are $\theta=0$ and $\theta=\pi/3$

16) What is the total area between the polar curves $r = 2\sin 3\theta$ and $r = 5\sin 3\theta$.



Area of region for 1 petal = Area (larger petal) − Area (smaller petal)

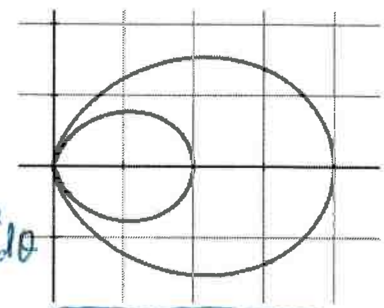
$$\text{Area} = \frac{1}{2}\int_0^{\pi/3}(5\sin 3\theta)^2\,d\theta - \frac{1}{2}\int_0^{\pi/3}(2\sin 3\theta)^2\,d\theta$$

Area = 5.4977

Area of all 3 petals $\rightarrow 3(5.4977) = \boxed{16.493}$

17)

The figure to the right shows the graphs of the polar curves $r = 2\cos^2\theta$ and $r = 4\cos^2\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Which of the following integrals gives the area of the region bounded between the two polar curves?



A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

B. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6\cos^4\theta\,d\theta$

C. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^4\theta\,d\theta$

D. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta$

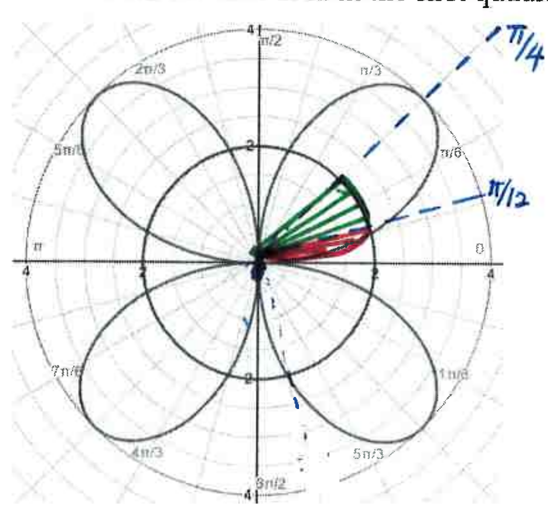
Area of Larger − Area of Smaller

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}[4\cos^2\theta]^2\,d\theta - \frac{1}{2}\int_{-\pi/2}^{\pi/2}[2\cos^2\theta]^2\,d\theta$$

$$\frac{1}{2}\int 16\cos^4\theta\,d\theta - \frac{1}{2}\int 4\cos^4\theta\,d\theta$$

$$8\int \cos^4\theta\,d\theta - 2\int \cos^4\,d\theta \rightarrow \boxed{\int_{-\pi/2}^{\pi/2} 6\cos^4\theta\,d\theta}$$

18) Find the total area in the first quadrant of the common interior of $r = 4\sin 2\theta$ and $r = 2$.



*intersection: $4\sin 2\theta = 2$

$\sin 2\theta = \frac{2}{4}$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \sin^{-1}\left(\frac{1}{2}\right)$

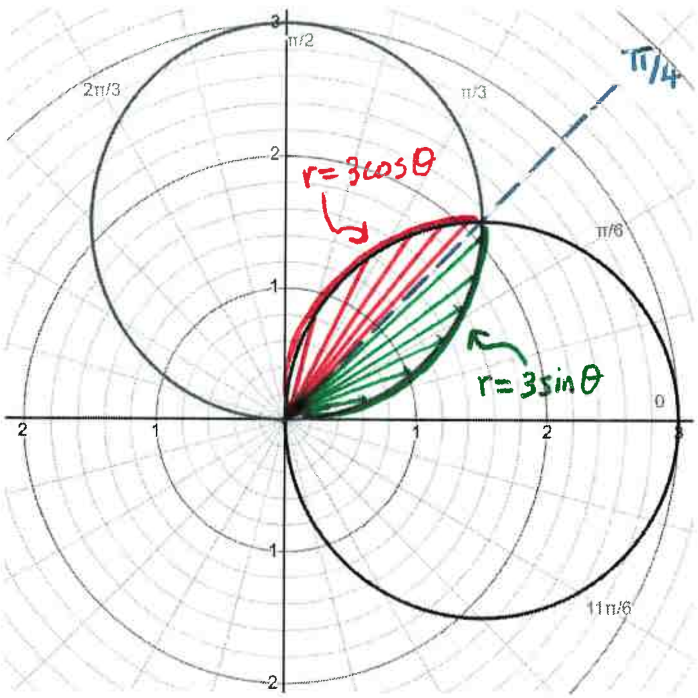
$2\theta = \pi/6$

$\theta = \frac{1}{2}\cdot\pi/6 = \frac{\pi}{12}$

$$2\left[\frac{1}{2}\int_0^{\pi/12}(4\sin 2\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{\pi/4}(2)^2\,d\theta\right]$$

$\boxed{\text{Area} = 2.457}$

19) Find the area of the common interior of the polar graphs $r = 3\cos\theta$ and $r = 3\sin\theta$.



*intersection:

$3\sin\theta = 3\cos\theta$

$\frac{3\sin\theta}{3\cos\theta} = \frac{3\cos\theta}{3\cos\theta}$

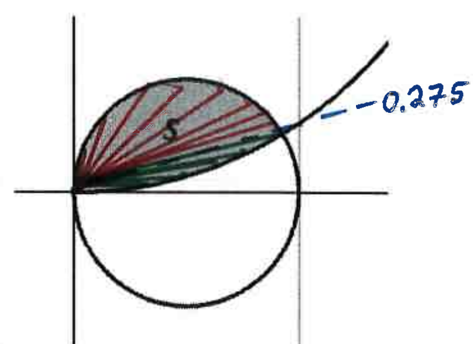
$\tan\theta = 1 \rightarrow \theta = \pi/4$

$$\frac{1}{2}\int_0^{\pi/4}[3\sin\theta]^2 d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}[3\cos\theta]^2 d\theta$$

$\boxed{\text{Area} = 1.284}$

20)

Let S be the region in the 1st Quadrant bounded above by the graph of the polar curve $r = \cos\theta$ and bounded below by the graph of the polar curve $r = \frac{7}{2}\theta$, as shown in the figure. The two curves intersect when $\theta = 0.275$. What is the area of S?



$$\text{Area} = \frac{1}{2}\int_0^{0.275}\left(\frac{7}{2}\theta\right)^2 d\theta + \frac{1}{2}\int_{0.275}^{\pi/2}[\cos\theta]^2 d\theta$$

$$0.04246 + 0.258613 \approx \boxed{0.301}$$

(Unit 9) **Parametric functions**: where x and y coordinates on a graph are given in terms of a third variable "t" : x=f(t) and y = g(t) are parametric equations and **t** is called the parameter.

*Eliminate the parameter, and use substitution to write rectangular equation.

*Rectangular equation only shows path of graph

*Parametric Equation tracks more info: Includes the **path, speed,** and **direction of graph**

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{y'(t)}{x'(t)} \quad \text{and} \quad \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

*Horizontal tangent occurs where $\frac{dy}{dt}=0$

*Vertical tangent occurs where $\frac{dx}{dt}=0$

*Beware of $\frac{0}{0}$, which is neither a horizontal nor vertical tangent

Parametric Arc Length: $(s) = \int_{t1}^{t2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Speed of particle: $\left|\vec{v}(t)\right|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

Distance of particle $\int_{t_1}^{t_2}|v|=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

*Distance of particle **IS** the parametric arc length

* $\int speed$ is the **total distance traveled** along the curve (arc length)

* $\int velocity$ is the **displacement** (net change in position where positives and negatives cancel)

* **Final Position = Initial Position + Displacement**

Arc Length:

Arc length (s) $=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

Parametric $s=\int_{t1}^{t2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ or $\int_{t1}^{t2}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

$ds^2 = dx^2 + dy^2$

Area of a Surface of Revolution in Parametric Form

Revolution about **x-axis**: $S=2\pi\int_{t_1}^{t_2}y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$ Revolution about **y-axis**: $S=2\pi\int_{t_1}^{t_2}x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$

(Ch. 10.4 – 10.5) Polar Equations: Ordered pairs are expressed as (r,θ) with θ as the independent variable.

r = distance from origin $\quad\theta$= directed angle from polar axis

origin is called the pole. $\quad$ x-axis is called the polar axis

Polar to Rectangular

$x = r\cos\theta$

$y = r\sin\theta$

Rectangular to Polar

$r=\sqrt{x^2+y^2}$

$\tan\theta=\frac{y}{x}$

Polar Derivatives:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{y'(\theta)}{x'(\theta)}$$

Region bounded by a Polar Curve:

Area of a circular sector: $A=\frac{1}{2}r^2\theta$

Polar Area Enclosed Region: $A=\int_a^b\frac{1}{2}r^2\,d\theta$

Arc Length Polar Curve: $s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+[r'(\theta)]^2}\,d\theta$

(Ch. 10.4) Special Polar Graphs

Circles: $r=a\cos\theta$ or $r=a\sin\theta$

*Traces out 1 rotation CCW from $[0,\pi]$

*coefficient **a** is the length of the diameter

*cosine graph symmetric to x-axis

*sine graph symmetric to y-axis

CCW = counter clockwise

Limacons: r = a ± b cos θ or r = a ± b sin θ $\quad(a>0, b>0)$

* Traces out 1 rotation Clockwise from $[0,2\pi]$

*constant + coefficient = outer radius

*constant – coefficient = inner radius

Graphs going through poles (origin):

Cardioid: **once**

Limacon with inner loop: **twice**

Dimpled Limacon: **none**

Rose Curves $r=a\cos(n\theta)$, $r=a\sin(n\theta)$

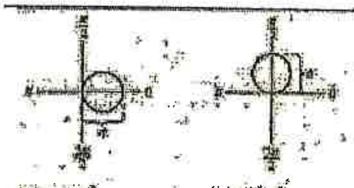
*coefficient **a** is the length of each petal

*If **n** is odd, then there are **n** petals on graph

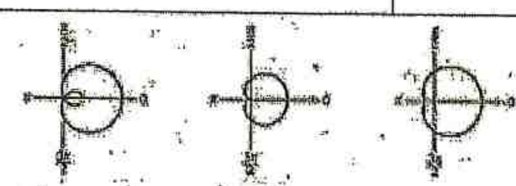
*If **n** is even, then there are **2n** petals on graph

*If **n** is odd, 1 rotation traces out CCW from $[0,\pi]$

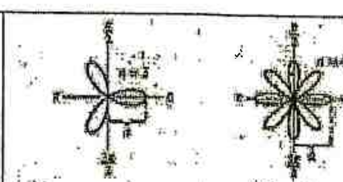
*If **n** is even, 1 rotation traces out CCW from $[0,2\pi]$



Circle Circle



Limacon with inner loop Cardioid (heart-shaped) Dimpled limacon



Rose curve Rose curve